Determining modes and nodes of the rotating Navier-Stokes equations

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Abstract: We analyse the long-term dynamics of the two-dimensional Navier–Stokes equations on a rotating sphere and the periodic β-plane, which can be considered as a planar approximation to the former. It was shown over fifty years ago that the Navier–Stokes equations can be described by a finite number of degrees of freedom, which can be quantified by, for example, the so-called determining modes and determining nodes. After considerable effort, it was shown that, independently of rotation, the number of determining modes and nodes both scale as the Grashof number $\mathcal{G}$, a non-dimensional parameter proportional to the forcing.

Using and extending recent results on the behaviour of the rotating Navier–Stokes equations, we prove under reasonable hypotheses that the number of determining modes is bounded by $c\,\mathcal{G}^{1/2} + \varepsilon^{1/2}M$, where $1/\varepsilon$ is the rotation rate and M depends on up to third derivatives of the forcing. Our bound on the number of determining nodes is slightly weaker, at $c\,\mathcal{G}^{2/3} + \varepsilon^{1/2}M$.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification. Chapters 3 and 5 are based on joint research with Dr. Djoko Wirosoetisno [1]. All other work is mine, unless explicitly stated otherwise within the text.

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How can we go from finity to infinity? Unless finity can become infinity we can never understand either. An individual never understands anyone else.

— from *The field of Zen* by D. T. Suzuki

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Chapter 1

Introduction

The incompressible Navier–Stokes equations describe the flow of a fluid and are used in applications such as weather prediction, modelling air flow around aircrafts and studying ocean currents. The equations are given by

$$\begin{aligned} \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p &= \mu \Delta \boldsymbol{v} + f_{\boldsymbol{v}}, \\ \nabla \cdot \boldsymbol{v} &= 0, \end{aligned} \tag{1.0.1}$$

where $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ is the fluid velocity at a point $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $n = 2$ or 3 and time $t \in \mathbb{R}$, p is the pressure, μ is the kinematic viscosity and $f_{\boldsymbol{v}}$ is the forcing. The existence and uniqueness of solutions to the three-dimensional case in general are still unknown; in contrast, these are well-known for the two-dimensional case ([2]–[4]). In this thesis we consider the case of $n = 2$ on a doubly periodic plane and the unit sphere.

It is often useful to consider (1.0.1) on a rotating frame, as this is a naturally arising situation in which we consider a fluid, for example with the effect of the earth's rotation on ocean currents and the atmosphere. By describing the earth's rotation by a constant vector Ω, the Coriolis parameter is given by $f = 2\Omega \sin\theta$, where θ is the latitude (defined to be 0 at the north pole). We frequently model a fluid flow on the surface of the earth by using a unit rotating sphere, on which the Rossby parameter becomes $\beta := \partial_y f = 2\Omega \cos\theta$. The rotating Navier–Stokes equations on the unit sphere thus read as

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{2}{\varepsilon} \cos\theta\, \boldsymbol{v}^{\perp} + \nabla p = \mu \Delta \boldsymbol{v} + f_{\boldsymbol{v}},$$
$$\nabla \cdot \boldsymbol{v} = 0. \tag{1.0.2}$$

A simpler method of modelling a fluid on the earth's surface is the β-plane approximation, which is useful for phenomena that occur on a scale much smaller than the domain itself [5]. On such scales, one can reasonably approximate the behaviour of the fluid by instead considering it as being on a tangent plane, so that it becomes convenient to use a Cartesian coordinate system $(x_1, x_2) = (x, y)$ instead. Using the β-plane approximation involves allowing the Coriolis parameter to vary linearly, so that we describe it near latitude θ_0 as

$$f = f_0 + \beta y,$$

where f_0 is the Coriolis parameter at θ_0, $\beta = \partial_y f$ is the gradient of f in latitude and y is the meridional distance from θ_0. Due to incompressibility, a constant rate of rotation does not affect the dynamics, i.e. without loss of generality we can consider the fluid at the equator. This gives $f_0 = 0$, with which we obtain the β-plane approximation of (1.0.1):

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \beta y \boldsymbol{v}^{\perp} + \nabla p = \mu \Delta \boldsymbol{v} + f_{\boldsymbol{v}}, \tag{1.0.3}$$

where $\boldsymbol{v}^{\perp} = (-v_2, v_1)$, with v_1 and v_2 denoting the x and y components of $\boldsymbol{v}$ respectively.

Physical intuition suggests that the flow of a fluid on a rotating plane or sphere would become more zonal with increasing rotation rate; there are numerical works [6] and analytical proofs that agree with this ([7], [8]). Notably, both on the β-plane and the sphere, a rotation rate scaling as βy will eventually make the non-zonal part $\tilde{\boldsymbol{v}}$ of the flow bounded by

$$|\nabla \times \tilde{\boldsymbol{v}}|_{L^2}^2 \quad \lesssim \quad O(\varepsilon). \tag{1.0.4}$$

In this thesis, we combine this result regarding the smallness of the non-zonal flow undergoing fast rotation, with determining modes and nodes.

It has been known for over half a century [9] that the two-dimensional Navier–Stokes equations can be described by a finite number of degrees of freedom. For periodic boundary conditions, an upper estimate on the Hausdorff dimension of the global attractor $\mathcal{A}$ was made by Constantin et al [10], giving

$$dim_H(\mathcal{A}) \le c\,\mathcal{G}^{2/3}\,(1+\log\mathcal{G})^{1/3}, \tag{1.0.5}$$

where $\mathcal{G} := |f_{\boldsymbol{v}}|_{L^2}/(\mu^2\kappa_0^2)$ is the Grashof number, a parameter used to describe how turbulent a flow is. A lower bound on the attractor dimension was given by Liu [11]:

$$c\,\mathcal{G}^{2/3} \le \dim_H(\mathcal{A}), \tag{1.0.6}$$

thereby proving that the aforementioned upper estimate is in fact sharp, up to a logarithm. For the rotational case with periodic boundary conditions, Al-Jaboori and Wirosoetisno [7] showed that with sufficiently large β, $dim_H(\mathcal{A}) = 0$.

The theory of determining modes was introduced by Foias and Prodi [9], to describe the number of degrees of freedom of the two-dimensional Navier–Stokes equations. One considers two solutions of (1.0.1), with the same viscosity μ:

$$\begin{gathered}\partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p = \mu\Delta\boldsymbol{v} + f_{\boldsymbol{v}}, \qquad (1.0.7)\\ \partial_t\boldsymbol{v}^{\#} + \boldsymbol{v}^{\#}\cdot\nabla\boldsymbol{v}^{\#} + \nabla p^{\#} = \mu\Delta\boldsymbol{v}^{\#} + f_{\boldsymbol{v}^{\#}},\end{gathered}$$

where $|(f_{\boldsymbol{v}} - f_{\boldsymbol{v}^{\#}})(t)|_{L^2} \to 0$ as $t \to \infty$. The rough idea is that when we consider the projection of the difference $\delta\boldsymbol{v} := \boldsymbol{v} - \boldsymbol{v}^{\#}$ to lower and higher wavenumbers (modes), if the difference in the lower modes converges to 0 after time, then the whole of $\delta\boldsymbol{v}$ will also eventually converge to 0. This will be made rigorous in the relevant chapters. Over the years, progress has been made in bounding the number of determining modes, or the minimal threshold wavenumber required to guarantee this convergence. Jones and Titi [12] proved that on the periodic plane, the number of determining modes scales as $\kappa \sim c\,\mathcal{G}^{1/2}$, which agrees with what is expected based on physical arguments ([13], [14]).

The idea of determining nodes was first introduced by Foias and Temam [15]. Similarly to determining modes, one considers the difference between two solutions of the Navier–Stokes equations (1.0.7). The approach here is that with the solutions being identical at a large enough (but finite) number of points, the difference in the solutions will again converge to 0 after sufficient time. Foias and Temam gave bounds on the maximal distance between these nodal points, depending only on μ, the domain and the forcing; others such as Jones and Titi [12] instead proved bounds on the total number of nodes required, which scales as $N \sim c\,\mathcal{G}$. Robinson and Friz [16] showed that the number of nodes is bounded from below by the fractal dimension of the global attractor:

$$\dim_f(\mathcal{A}) < c\,N. \tag{1.0.8}$$

We are unaware of results in the opposite direction, i.e. lower bounds on the attractor dimension in terms of the number of nodes, which would be useful in practice when combined with our results, as listed below.

In this thesis, we prove that under a sufficiently fast differential rotation, the number of determining modes and nodes are reduced for both the torus and unit sphere, when compared with the general non-rotating case. The structure of the thesis is as follows. After listing a collection of definitions and general inequalities in Chapter 2, the main content starts in Chapter 3, where we derive bounds on the number of modes over the torus. We then prove similar bounds on the number of modes over the sphere in Chapter 4, after stating and/or showing the necessary spherical equivalents of results from the previous chapter.

We formally introduce the concept of determining nodes in Chapter 5, followed by an auxiliary lemma from Jones and Titi [12], recast in a more suitable form for generalisation to S^2 later in Chapter 6. Using these, we prove our improved bounds on the nodes. Finally, in Chapter 6, we derive our bounds on the nodes over the sphere. For this we require a spherical analogue of the aforementioned auxiliary lemma, which we obtain via an icosahedral triangulation of the sphere.

We note that even though we require different tools and auxiliary results to prove our theorems over the torus and the sphere, the results themselves are of the same order over each domain. This suggests that even though there are technical differences between the domains, the rotating 2D Navier–Stokes equations behave in a fundamentally similar way over them.

Chapter 2

Background

The purpose of this chapter is to collect definitions and well-known inequalities used throughout the thesis for reference. The majority of the results hold over both the torus and the sphere; some definitions and results on the sphere have been deferred to Chapter 4.

2.1 Notation

For this thesis, we work with either a torus $\mathbb{T}^2 := [0, L] \times [-L/2, L/2]$ or the unit sphere $S^2 := \{(\theta, \phi) : \theta \in [0, \pi], \phi \in [0, 2\pi)\}$.

Let Ω be either $\mathbb{T}^2$ or S^2. For $1 \leq p \leq \infty$, we denote by $L^p(\Omega)$ the Lebesgue space, consisting of the space of Lebesgue measurable functions $u : \Omega \to \mathbb{R}^n$ such that

$$\int_\Omega |u(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} < \infty,$$

with respect to the corresponding metric. When $1 \leq p < \infty$, $L^p(\Omega)$ is also a Banach space when equipped with the norm

$$|u|_{L^p(\Omega)} := \left(\int_\Omega |u(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p}.$$

For $p = \infty$, $L^\infty(\Omega)$ consists of functions on Ω that are measurable and essentially bounded. It is also a Banach space, when equipped with the norm

$$
\begin{aligned}
|u|_{L^\infty(\Omega)} &:= \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |u(\boldsymbol{x})| \\
&= \inf \{\sup_{\boldsymbol{x}\in\Lambda} |u(\boldsymbol{x})| : \Lambda \subset \overline{\Omega},\ \ \Omega\backslash\Lambda \ \text{ has zero measure}\}.
\end{aligned}
$$

In the special case of $p = 2$, $L^2(\Omega)$ is also a Hilbert space, with inner product and norm defined by

$$
\begin{aligned}
(u, v)_{L^2(\Omega)} &:= \int_\Omega u(\boldsymbol{x})\overline{v(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}, \\
|u|_{L^2(\Omega)} &:= (u, u)^{1/2}_{L^2(\Omega)},
\end{aligned}
$$

where $\overline{v}$ denotes the complex conjugate of v. When the domain of integration is clear and no confusion would arise, we may write $|\cdot|_p := |\cdot|_{L^p}$, $|\cdot| := |\cdot|_{L^2}$ and $(\cdot,\cdot) := (\cdot,\cdot)_{L^2}$ for conciseness.

The Sobolev space $H^s(\Omega)$ consists of functions with derivatives up to order s lying in $L^2(\Omega)$:

$$
H^s(\Omega) := \{u : D^m u \in L^2(\Omega), \quad \forall\, |m| \le s\},
$$

where

$$
\begin{aligned}
D^m u &:= \partial^{m_1}_{x_1}\partial^{m_2}_{x_2} u, \\
|m| &:= m_1 + m_2, \qquad\qquad m_1, m_2 \ge 0.
\end{aligned}
$$

The inner product and norms are defined by

$$
\begin{aligned}
(u, v)_{H^s(\Omega)} &= \sum_{|m|\le s} (D^m u, \overline{D^m v})_{L^2(\Omega)}, \\
|u|_{H^s(\Omega)} &:= (u, u)^{1/2}_{H^s(\Omega)},
\end{aligned}
$$

which is equivalent to the norm defined by

$$
|u|^2_{H^s_*(\Omega)} := |u|^2_{L^2(\Omega)} + \sum_{|m|=s} (D^m u, \overline{D^m u})_{L^2(\Omega)}. \tag{2.1.1}
$$

Furthermore, for u satisfying

$$\int_\Omega u \, \mathrm{d}\boldsymbol{x} = 0, \tag{2.1.2}$$

(2.1.1) is also equivalent to

$$|u|^2_{H^s_*(\Omega)} \sim \sum_{|m|=s} (D^m u, \overline{D^m u})_{L^2(\Omega)}, \tag{2.1.3}$$

due to the Fourier coefficient $u_{\boldsymbol{k}} = u_{(0,0)}$ (for $\Omega = \mathbb{T}^2$) or spherical harmonic coefficient $u_{lm} = u_{00}$ (introduced in Chapter 4, when $\Omega = S^2$) being zero by definition. Functions we consider in this thesis all satisfy (2.1.2) over the respective domain and hence (2.1.3) is equivalent to the H^s norm; we thus abuse notation slightly and denote instead by $H^s(\Omega)$,

$$H^s(\Omega) = \{u : D^m u \in L^2(\Omega), \quad \forall\, |m| = s\}. \tag{2.1.4}$$

As a useful (dimensionless) parameter to describe how turbulent a flow is, we define the generalised Grashof number $\mathcal{G}$ by

$$\mathcal{G} := \frac{|f_{\boldsymbol{v}}|_{L^2(\Omega)}}{\mu^2 \kappa_0^2}, \tag{2.1.5}$$

where μ is the kinematic viscosity, $\kappa_0 = \kappa_0(\Omega)$ is the Poincaré constant (see Lemma 4). Using higher derivatives of $f_{\boldsymbol{v}}$, we define "higher Grashof numbers" by

$$\mathcal{G}_m := \frac{|\nabla^m f_{\boldsymbol{v}}|_{L^2(\Omega)}}{(\mu\kappa_0)^{2-m}}, \tag{2.1.6}$$

where the denominator ensures that $\mathcal{G}_m$ is dimensionless for all m. We note that $\mathcal{G}_0 = \mathcal{G}$ exactly, with this definition.

Throughout the thesis, unnumbered constants c denote dimensionless constants that may change value from one use to the next. We will also drop all dimensional quantities except length; it will thus become convenient to define the dimensionless parameter $\nu_0 := \mu\kappa_0^2$.

2.2 Preliminary inequalities

We make extensive use of several well-known inequalities, which we collectively state here for convenience.

Lemma 1 (Young's inequality). *Suppose a, $b \geq 0$ and $1 \leq p, q \leq \infty$ are such that $1/p + 1/q = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.2.1}$$

Lemma 2 (Hölder's inequality). *Suppose $f \in L^p(\mathbb{T}^2)$ and $g \in L^q(\mathbb{T}^2)$, where $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Then $fg \in L^1(\mathbb{T}^2)$, and*

$$|fg|_{L^1(\mathbb{T}^2)} \leq |f|_{L^p(\mathbb{T}^2)}\, |g|_{L^q(\mathbb{T}^2)}. \tag{2.2.2}$$

Lemma 3 (Hölder's inequality on the sphere). *Suppose $f \in L^p(S^2)$ and $g \in L^q(S^2)$, where $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Then $fg \in L^1(S^2)$, and*

$$|fg|_{L^1(S^2)} \leq |f|_{L^p(S^2)}\, |g|_{L^q(S^2)}. \tag{2.2.3}$$

Lemma 4 (Poincaré's inequality). *Suppose $u \in H^1(X)$ for bounded X. Then there exists $\kappa_0 > 0$, depending only on X, such that*

$$\kappa_0 |u|_{L^2(X)} \leq |\nabla u|_{L^2(X)}. \tag{2.2.4}$$

For $X = \mathbb{T}^2$ and u such that $\int_{\mathbb{T}^2} u = 0$, the Poincaré constant κ_0 is given by

$$\begin{aligned}\kappa_0 :=& \inf_u \frac{|\nabla u|_{L^2}}{|u|_{L^2}} \\ =& \inf_u \Big(\big(\sum_{\boldsymbol{k} \in \mathbb{Z}_L} |\boldsymbol{k}|^2 |u_{\boldsymbol{k}}|^2 \big) / \big(\sum_{\boldsymbol{k} \in \mathbb{Z}_L} |u_{\boldsymbol{k}}|^2 \big) \Big)^{1/2} = 2\pi/L,\end{aligned}$$

where $\mathbb{Z}_L := \{(2\pi l_1/L, 2\pi l_2/L) : (l_1, l_2) \in \mathbb{Z}^2\}$ and $u_{\boldsymbol{k}}$ are the Fourier coefficients of u. The corresponding Poincaré constant for $X = S^2$ will be computed in Chapter 4, after a suitable expansion of $v \in L^2(S^2)$ into its harmonics is introduced.

Lemma 5 (Agmon's inequality in 1D). *Suppose $u \in L^\infty([0, 1]) \cap H^1([0, 1])$. Then*

$$|u|_{L^\infty([0,1])} \leq c_1\, |u|^{1/2}_{L^2([0,1])} |\nabla u|^{1/2}_{L^2([0,1])}, \tag{2.2.5}$$

where $c_1 = 1/\pi + 2$.

Proof. Let $\gamma, x \in [0,1]$. By the fundamental theorem of calculus,

$$u^2(x) = u^2(\gamma) + \int_\gamma^x (u^2(y))' \, \mathrm{d}y = u^2(\gamma) + 2\int_\gamma^x u(y)u'(y) \, \mathrm{d}y.$$

We bound this from above as

$$\begin{aligned} u^2(x) &= u^2(\gamma) + 2\int_\gamma^x u(y)u'(y) \, \mathrm{d}y \\ &\leq u^2(\gamma) + 2\int_\gamma^x |u(y)u'(y)| \, \mathrm{d}y \\ &\leq u^2(\gamma) + 2|uu'|_{L^1([0,1])} \\ &\leq u^2(\gamma) + 2|u|_{L^2([0,1])}|u'|_{L^2([0,1])} && \text{by Hölder.} \end{aligned}$$

Integrating both sides from $\gamma = 0$ to 1 gives

$$\begin{aligned} u^2(x) &\leq \int_0^1 u^2(\gamma) \, \mathrm{d}\gamma + 2|u|_2|u'|_2 \\ &= |u|_2^2 + 2|u|_2|u'|_2 \\ &\leq \kappa_0^{-1}|u|_2|u'|_2 + 2|u|_2|u'|_2 && \text{by Poincaré} \\ &= \left(\kappa_0^{-1} + 2\right)|u|_2|u'|_2. \end{aligned}$$

The Poincaré constant for $[0,1]$ is bounded by $1/\pi$ [17], so taking the maximum value of the left side gives

$$\begin{aligned} |u|_\infty^2 &\leq \left(\kappa_0^{-1} + 2\right)|u|_2|u'|_2 \\ &\leq (\pi^{-1} + 2)|u|_2|u'|_2. \end{aligned}$$

□

We note that this one-dimensional version of the more well-known Agmon's inequalities in $\mathbb{R}^2$ or $\mathbb{R}^3$ is particularly useful when we consider zonal (i.e. independent of x) functions. This will be made clear in the relevant chapters.

Lemma 6 (Agmon's inequality in 2D)**.** *Suppose* $u \in L^\infty(\mathbb{T}^2) \cap H^1(\mathbb{T}^2)$. *Then there* *exists a constant* $c_2(\mathbb{T}^2)$ *such that*

$$|u|_{L^\infty(\mathbb{T}^2)} \leq c_2\, |u|^{1/2}_{L^2(\mathbb{T}^2)} |\Delta u|^{1/2}_{L^2(\mathbb{T}^2)}. \tag{2.2.6}$$

There are many different interpolation inequalities that are useful for bounding functions in Sobolev spaces. We will only be requiring the following for the purposes of this thesis:

Lemma 7 (Ladyzhenskaya's inequality)**.** *Suppose* $u \in H^1(\Omega)$*, where* $\Omega = \mathbb{T}^2$ *or* S^2*. Then there exists a constant* $c_3(\Omega)$ *such that*

$$|u|_{L^4(\Omega)} \leq c_3\, |u|^{1/2}_{L^2(\Omega)} |\nabla u|^{1/2}_{L^2(\Omega)}. \tag{2.2.7}$$

We will make extensive use of the following Gronwall-type inequality ([13], [18]):

Lemma 8. *Let* ρ *be a locally integrable real function on* $(0, \infty)$ *such that*

$$\liminf_{t\to\infty} \int_t^{t+1} \rho(\tau)\,\mathrm{d}\tau > 0, \tag{2.2.8}$$

$$\limsup_{t\to\infty} \int_t^{t+1} \rho^-(\tau)\,\mathrm{d}\tau < \infty, \tag{2.2.9}$$

where $\rho^- := \max\{-\rho, 0\}$*. Also, let* σ *be a real locally integrable function on* $(0,\infty)$ *such that*

$$\lim_{t\to\infty} \int_t^{t+1} \sigma^+(\tau)\,\mathrm{d}\tau = 0, \tag{2.2.10}$$

where $\sigma^+ := \max\{\sigma, 0\}$*. Suppose* ξ *is an absolutely continuous non-negative function on* $(0, \infty)$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi + \rho\xi \leq \sigma \qquad \textit{almost everywhere on } (0,\infty).$$

Then $\xi(t) \to 0$ *as* $t \to \infty$*.*

We also often use the following integral inequality.

Lemma 9. *Let* $\nu > 0$ *be fixed and* $u(t) \geq 0$*. Suppose that for any* $t \geq 1$*, we have*

$$\int_0^t u(\tau)\mathrm{e}^{\nu(\tau - t)}\,\mathrm{d}\tau \leq M.$$

Then for any $t > 0$,

$$\int_t^{t+1} u(\tau)\,\mathrm{d}\tau \le \int_t^{t+1} \mathrm{e}^{\nu(\tau-t)} u(\tau)\,\mathrm{d}\tau \le \int_0^{t+1} \mathrm{e}^{\nu(\tau-t)} u(\tau)\,\mathrm{d}\tau \le \mathrm{e}^{\nu} M. \tag{2.2.11}$$

Chapter 3

Determining modes on the periodic β-plane

In this chapter we prove our main result concerning the number of determining modes on the rotating torus $\mathbb{T}^2 = [0, L] \times [-L/2, L/2]$. We outline the existing theory on the modes for the Navier–Stokes equations in Section 3.1.2, which will become the basis for Chapter 4 also. We then introduce the zonal and non-zonal components of the vorticity in Section 3.2, as well as citing a useful control on the non-zonal vorticity from [7]. Combining these two elements together, we state and prove our result on the number of determining modes in Section 3.4.

Whilst the β-plane is an approximation of the sphere, we will see later in Chapter 4 that our results on the periodic plane and the sphere are of the same order.

3.1 Statement of the problem

We derive the vorticity form of the Navier–Stokes equations below, which is both more convenient for our purposes and more useful for numerical simulations. We recall that when there is no ambiguity, we may write $|\cdot|_p = |\cdot|_{L^p}$, $|\cdot| = |\cdot|_{L^2}$ and $(\cdot, \cdot) = (\cdot, \cdot)_{L^2}$.

3.1.1 Vorticity form

We recall the β-plane approximation of the Navier–Stokes equations:

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \beta y \boldsymbol{v}^\perp + \nabla p = \mu \Delta \boldsymbol{v} + f_{\boldsymbol{v}}, \tag{3.1.1}$$
$$\nabla \cdot \boldsymbol{v} = 0.$$

By Hodge's decomposition theorem (see [19]), $\boldsymbol{v}$ can be written as

$$\boldsymbol{v} = \nabla \varphi + \nabla^\perp \psi + \boldsymbol{H}, \tag{3.1.2}$$

where φ, ψ are scalars and $\boldsymbol{H}$ is a (curl-free and divergence-free) harmonic vector field. Since $\nabla \cdot \boldsymbol{v} = 0$, taking the divergence of (3.1.2) gives

$$\begin{aligned} 0 = \nabla \cdot \boldsymbol{v} &= \nabla \cdot \nabla \varphi + \nabla \cdot \nabla^\perp \psi + \nabla \cdot \boldsymbol{H} \\ &= \Delta \varphi. \end{aligned} \tag{3.1.3}$$

Expanding $\Delta\varphi$ in Fourier series leads to

$$\Delta \varphi(\boldsymbol{x}, t) = -\sum_{\boldsymbol{k} \in \mathbb{Z}_L} |\boldsymbol{k}|^2 \varphi_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = 0, \tag{3.1.4}$$

where we recall $\mathbb{Z}_L := \{(2\pi l_1/L, 2\pi l_2/L) : (l_1, l_2) \in \mathbb{Z}^2\}$. This implies that $\varphi_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \neq \boldsymbol{0}$, thus

$$\varphi(\boldsymbol{x}, t) = \sum_{\boldsymbol{k} \in \mathbb{Z}_L} \varphi_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = \varphi_{(0,0)}(t), \tag{3.1.5}$$

implying that φ is constant in $\boldsymbol{x}$, i.e. $\nabla\varphi = 0$. Hence (3.1.2) becomes

$$\boldsymbol{v} = \nabla^\perp \psi + \boldsymbol{H}. \tag{3.1.6}$$

There are exactly two independent harmonic vector fields in $\mathbb{T}^2$, which we can take to be the constant vector fields $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ ([19]). Without loss of generality (see [20]), we can take $\boldsymbol{H} = 0$, which is equivalent to

$$\int_{\mathbb{T}^2} \boldsymbol{v}(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} = 0 \qquad \forall t \geq 0, \tag{3.1.7}$$

which we assume. Consequently, we also require

$$\int_{\mathbb{T}^2} f_{\boldsymbol{v}}(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = 0 \qquad \forall t \geq 0. \tag{3.1.8}$$

With these assumptions, $\boldsymbol{v}$ is determined uniquely from ω.

Requiring compatibility of the $\beta y \boldsymbol{v}^{\perp}$ term over the periodic domain implies the following natural symmetries, which we also assume:

$$v_1(x, -y, t) = v_1(x, y, t), \tag{3.1.9}$$

$$v_2(x, -y, t) = -v_2(x, y, t), \tag{3.1.10}$$

where we recall that v_1 denotes the x component of $\boldsymbol{v}$ and similarly for v_2. Together with periodicity, (3.1.10) implies that

$$v_2(x, -L/2, t) = -v_2(x, L/2, t) = 0. \tag{3.1.11}$$

We impose analogous symmetries on $f_{\boldsymbol{v}}$, which we assume to be time-independent:

$$f_{v_1}(x, -y) = f_{v_1}(x, y), \tag{3.1.12}$$

$$f_{v_2}(x, -y) = -f_{v_2}(x, y). \tag{3.1.13}$$

With these assumptions, we take the curl $\nabla^{\perp}\cdot$ of (3.1.1), where $\nabla^{\perp}\cdot\boldsymbol{v} = \partial_x v_2 - \partial_y v_1 =: \omega$ is the scalar vorticity:

$$\nabla^{\perp}\cdot\partial_t\boldsymbol{v} + \nabla^{\perp}\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \beta\nabla^{\perp}\cdot(y\boldsymbol{v}^{\perp}) + \nabla^{\perp}\cdot\nabla p = \mu\nabla^{\perp}\cdot(\Delta\boldsymbol{v}) + \nabla^{\perp}\cdot f_{\boldsymbol{v}}. \tag{3.1.14}$$

The first term becomes

$$\nabla^{\perp}\cdot\partial_t\boldsymbol{v} = \partial_x(\partial_t\boldsymbol{v})_2 - \partial_y(\partial_t\boldsymbol{v})_1 = \partial_t(\partial_x v_2 - \partial_y v_1) = \partial_t\omega. \tag{3.1.15}$$

The second term of (3.1.14) is given by

$$\begin{aligned}
\nabla^{\perp}\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v}) &= \nabla^{\perp}\cdot(v_1\partial_x + v_2\partial_y, (v_1, v_2)) \\
&= \partial_x(v_1\partial_x v_2 + v_2\partial_y v_2) - \partial_y(v_1\partial_x v_1 + v_2\partial_y v_1) \\
&= -\partial_y v_1(\partial_x v_1 + \partial_y v_2) + \partial_x v_2(\partial_x v_1 + \partial_y v_2)
\end{aligned}$$

$$
\begin{aligned}
&\quad + v_1\partial_x(\partial_x v_2 - \partial_y v_1) + v_2\partial_y(\partial_x v_2 - \partial_y v_1)\\
&= v_1\partial_x(\partial_x v_2 - \partial_y v_1) + v_2\partial_y(\partial_x v_2 - \partial_y v_1)\\
&\qquad\qquad\qquad\qquad \text{(since } \partial_x v_1 + \partial_y v_2 = 0 \text{ by (3.1.1))}\\
&= \partial_x\psi\,\partial_y\omega - \partial_y\psi\,\partial_x\omega =: \partial(\psi,\omega), \qquad (3.1.16)
\end{aligned}
$$

where $\psi := \Delta^{-1}\omega$ is the streamfunction defined uniquely by $\int_{\mathbb{T}^2}\psi = 0$. The Jacobian $\partial(\cdot,\cdot)$ has the properties that

$$
\begin{aligned}
(\partial(a,b),b) &= \int_{\mathbb{T}^2}(\partial_x a\,\partial_y b - \partial_y a\,\partial_x b)\,b\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\mathbb{T}^2}(-\partial_y(a\,\partial_x b) + a\,\partial^2_{xy}b + \partial_x(a\,\partial_y b) - a\,\partial^2_{xy}b)\,b\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\mathbb{T}^2}\Big(-\partial_y(a\,\partial_x b) - \partial_x(a\,\partial_y b)\Big)b\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\mathbb{T}^2}(-a\,\partial_x b\,\partial_y b + a\,\partial_y b\,\partial_x b)\,\mathrm{d}\boldsymbol{x} \qquad \text{by integration by parts}\\
&= 0, \qquad (3.1.17)
\end{aligned}
$$

and

$$
\begin{aligned}
(\partial(a,b),c) &= \int_{\mathbb{T}^2}(\partial_x a\,\partial_y b - \partial_y a\,\partial_x b)\,c\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\mathbb{T}^2}\Big(-\partial_y(a\,\partial_x b) + \partial_x(a\,\partial_y b)\Big)\,c\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\mathbb{T}^2}(a\,\partial_x b\,\partial_y c - a\,\partial_y b\,\partial_x c)\,\mathrm{d}\boldsymbol{x} = (\partial(b,c),a)\\
&= (\partial(c,a),b) \qquad \text{by symmetry} \qquad (3.1.18)
\end{aligned}
$$

for all real a, b and c such that their integrals over $\mathbb{T}^2$ vanish and the expressions above are defined. To compute the third term of (3.1.14), we first replace y by the periodic extension of

$$
Y(y) = \begin{cases} 1 & \text{if } y = -L/2\\ y & \text{otherwise.}\end{cases}
$$

Taking the (distributional) curl of $Y\boldsymbol{v}^\perp$ gives

$$
\nabla^\perp\cdot(Y\boldsymbol{v}^\perp) = Y\nabla^\perp\cdot\boldsymbol{v}^\perp + \boldsymbol{v}^\perp\cdot\nabla^\perp Y = Y\nabla\cdot\boldsymbol{v} + \boldsymbol{v}^\perp\cdot\nabla^\perp Y = v_2 Y', \qquad (3.1.19)
$$

where $Y'(y) = 1 - L\delta(y - L/2)$ is the distributional derivative, with δ being the Dirac distribution, and the third equality follows from (3.1.1b). We note that (3.1.11) implies that $Y'(L/2)\, v_2(x, L/2, t) = 0$, so we can replace $v_2 Y'$ in (3.1.19) by v_2. Thus, the third term of (3.1.14) becomes

$$\frac{\kappa_0}{\varepsilon}\, v_2 = \frac{\kappa_0}{\varepsilon}\, \partial_x \psi, \tag{3.1.20}$$

where $\kappa_0/\varepsilon := \beta$. The pressure term of (3.1.14) becomes, by the properties of the curl and gradient,

$$\nabla^\perp \cdot \nabla p = 0. \tag{3.1.21}$$

The first term on the right hand side of (3.1.14) becomes

$$\mu \nabla^\perp \cdot (\Delta \boldsymbol{v}) = \mu \Delta \nabla^\perp \cdot \boldsymbol{v} = \mu \Delta \omega, \tag{3.1.22}$$

due to the commutativity of the Laplacian and the curl. Finally, we define the forcing on vorticity by

$$f := \nabla^\perp \cdot f_{\boldsymbol{v}} = \partial_x f_{v_2} - \partial_y f_{v_1}, \tag{3.1.23}$$

which inherits the time-independence of $f_{\boldsymbol{v}}$. Putting all these terms together, we obtain the vorticity form of the two-dimensional β-plane approximation of the Navier–Stokes equations:

$$\partial_t \omega + \partial(\psi, \omega) + \frac{\kappa_0}{\varepsilon} \partial_x \psi = \mu \Delta \omega + f. \tag{3.1.24}$$

We also note that due to the property of the curl, (3.1.7) implies

$$\int_{\mathbb{T}^2} \omega(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} = 0 \qquad \forall t \geq 0. \tag{3.1.25}$$

3.1.2 Theory of determining modes

In this section, we formally introduce and define the determining modes of the Navier–Stokes equations. Having derived the vorticity form, we now consider two solutions ω, $\omega^\sharp$ (with corresponding streamfunctions ψ, $\psi^\sharp$) of (3.1.24) with the same forcing and possibly different initial conditions:

$$\partial_t\omega + \partial(\psi,\omega) + \frac{\kappa_0}{\varepsilon}\,\partial_x\psi = \mu\Delta\omega + f, \tag{3.1.26}$$

$$\partial_t\omega^\sharp + \partial(\psi^\sharp,\omega^\sharp) + \frac{\kappa_0}{\varepsilon}\,\partial_x\psi^\sharp = \mu\Delta\omega^\sharp + f. \tag{3.1.27}$$

We note that our assumption on the forcing is slightly stronger than that made by Foias and Temam, in that we assume the forcings are equal (i.e. $f_{\boldsymbol{v}} = f_{\boldsymbol{v}^\sharp}$) rather than $\lim_{t\to\infty}|(f_{\boldsymbol{v}} - f_{\boldsymbol{v}^\sharp})(t)| = 0$. Qualitatively this does not make a difference, the general case being a straight forward extension; we have made the assumption purely for simplicity. By defining $\delta\omega := \omega - \omega^\sharp$ and $\delta\psi := \psi - \psi^\sharp$, we note that

$$\begin{aligned}
\partial(\psi,\omega) - \partial(\psi^\sharp,\omega^\sharp) &= \partial_x\psi\,\partial_y\omega - \partial_y\psi\,\partial_x\omega - \partial_x\psi^\sharp\partial_y\omega^\sharp + \partial_y\psi^\sharp\partial_x\omega^\sharp \\
&= \partial_x\psi^\sharp\partial_y\omega - \partial_x\psi^\sharp\partial_y\omega^\sharp - \partial_y\psi^\sharp\partial_x\omega + \partial_y\psi^\sharp\partial_x\omega^\sharp + \partial_x\psi\,\partial_y\omega \\
&\quad - \partial_x\psi^\sharp\partial_y\omega - \partial_y\psi\,\partial_x\omega + \partial_y\psi^\sharp\partial_x\omega \\
&= \partial_x\psi^\sharp\partial_y\delta\omega - \partial_y\psi^\sharp\partial_x\delta\omega + \partial_x\delta\psi\,\partial_y\omega - \partial_y\delta\psi\,\partial_x w \\
&= \partial(\psi^\sharp,\delta\omega) + \partial(\delta\psi,\omega).
\end{aligned}$$

Hence subtracting (3.1.27) from (3.1.26) gives

$$\partial_t\delta\omega + \partial(\psi^\sharp,\delta\omega) + \partial(\delta\psi,\omega) + \frac{\kappa_0}{\varepsilon}\,\partial_x\delta\psi = \mu\Delta\delta\omega. \tag{3.1.28}$$

We expand $\delta\omega$ in terms of its Fourier coefficients $\delta\omega_{\boldsymbol{k}}$:

$$\delta\omega(\boldsymbol{x},t) = \sum_{\boldsymbol{k}\in\mathbb{Z}_L} \delta\omega_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.1.29}$$

where we recall $\mathbb{Z}_L := \{(2\pi l_1/L, 2\pi l_2/L) : (l_1,l_2) \in \mathbb{Z}^2\}$. By fixing a threshold wavenumber $\kappa \geq \kappa_0$, we define P_κ as the L^2 projection to lower wavenumbers:

$$\delta\omega^<(\boldsymbol{x},t) := \mathsf{P}_\kappa\delta\omega(\boldsymbol{x},t) := \sum_{|\boldsymbol{k}|\leq\kappa} \delta\omega_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.1.30}$$

and the projection to higher modes by

$$\delta\omega^>(\boldsymbol{x},t) := \delta\omega(\boldsymbol{x},t) - \delta\omega^<(\boldsymbol{x},t) = \sum_{|\boldsymbol{k}|>\kappa} \delta\omega_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{3.1.31}$$

Using these definitions, we obtain the following Poincaré-type inequalities:

$$|\delta\omega^{>}|_{L^2}^2 = \sum_{|\boldsymbol{k}|>\kappa} |\delta\omega_{\boldsymbol{k}}|^2 \leq \sum_{|\boldsymbol{k}|>\kappa} (|\boldsymbol{k}|/\kappa)^2\, |\delta\omega_{\boldsymbol{k}}|^2 = \frac{1}{\kappa^2}|\nabla\delta\omega^{>}|_{L^2}^2, \tag{3.1.32}$$

and an inequality in the opposite direction,

$$|\nabla\delta\omega^{<}|_{L^2}^2 = \sum_{|\boldsymbol{k}|\leq\kappa} |\boldsymbol{k}|^2\, |\delta\omega_{\boldsymbol{k}}|^2 \leq \sum_{|\boldsymbol{k}|\leq\kappa} \kappa^2 |\delta\omega_{\boldsymbol{k}}|^2 = \kappa^2 |\delta\omega^{<}|_{L^2}^2. \tag{3.1.33}$$

It was shown by Foias and Prodi [9] that if one takes large enough κ, the behaviour of the (non-rotational) two-dimensional Navier–Stokes equations (1.0.1) can essentially be *determined* by the behaviour of the lower "modes", in the sense that if $|\mathsf{P}_\kappa \delta\omega(t)|_{L^2} \to 0$ as $t \to \infty$, then $|\delta\omega(t)|_{L^2} \to 0$ also. Manley and Trève ([13] [14]) conjectured that, based on physical arguments, the minimum number of determining modes in the general case scales as $\kappa/\kappa_0 \sim \mathcal{G}_0^{1/2}$. Jones and Titi's later result [12] agrees with this:

Theorem 10 (Jones and Titi '93)**.** *Suppose $\delta\omega$ satisfies* (3.1.28)*. There exists an absolute constant c_4 such that if*

$$\kappa/\kappa_0 \geq c_4\, \mathcal{G}_0^{1/2}, \tag{3.1.34}$$

then

$$\lim_{t\to\infty} |\mathsf{P}_\kappa \delta\omega(t)|_{L^2(\mathbb{T}^2)} = 0 \qquad \textit{implies} \qquad \lim_{t\to\infty} |\delta\omega(t)|_{L^2(\mathbb{T}^2)} = 0. \tag{3.1.35}$$

Our aim for this chapter is to obtain an improved bound on this existing result, making use of the regularising effect of rotation on the dynamics.

3.2 Zonal and non-zonal components of the vorticity

In order to improve bounds on the number of determining modes, it can be helpful to separate the vorticity into its zonal and non-zonal components, which we define by

$$\bar{\omega}(y,t) := \frac{1}{L}\int_0^L \omega(x,y,t)\,\mathrm{d}x, \tag{3.2.1}$$

$$\tilde{\omega}(x,y,t) := \omega(x,y,t) - \bar{\omega}(y,t). \tag{3.2.2}$$

We also express these in Fourier space as

$$\bar{\omega}(y,t) = \sum_{k_1=0} \omega_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3.2.3}$$

$$\tilde{\omega}(\boldsymbol{x},t) = \sum_{k_1\neq 0} \omega_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{3.2.4}$$

For convenience and consistency, we write

$$\bar{\omega}_{\boldsymbol{k}} = \begin{cases} \omega_{\boldsymbol{k}} & k_1 = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.5}$$

and

$$\tilde{\omega}_{\boldsymbol{k}} = \begin{cases} 0 & k_1 = 0 \\ \omega_{\boldsymbol{k}} & \text{otherwise.} \end{cases} \tag{3.2.6}$$

Thus $\bar{\omega}$ and $\tilde{\omega}$ are orthogonal in H^m for $m = 1, 2, \cdots$:

$$\begin{aligned}(\bar{\omega},\tilde{\omega})_{H^m} &= \sum_{\boldsymbol{k}} |\boldsymbol{k}|^m\,\bar{\omega}_{\boldsymbol{k}}\,|\boldsymbol{k}|^m\overline{\tilde{\omega}_{\boldsymbol{k}}} = \sum_{k_1=0} |\boldsymbol{k}|^m\,\bar{\omega}_{\boldsymbol{k}}\,|\boldsymbol{k}|^m\overline{\tilde{\omega}_{\boldsymbol{k}}} + \sum_{k_1\neq 0} |\boldsymbol{k}|^m\,\bar{\omega}_{\boldsymbol{k}}\,|\boldsymbol{k}|^m\overline{\tilde{\omega}_{\boldsymbol{k}}} \\ &= 0. \end{aligned} \tag{3.2.7}$$

We note that by definition, $\partial_x\bar{\omega} = 0$, which is particularly useful when we consider that

$$\partial_x U = \partial_x V = 0 \qquad \text{implies} \qquad \partial(U,V) = \partial_x U\,\partial_y V - \partial_y U\,\partial_x V = 0 \tag{3.2.8}$$

for any U, V such that the expression is defined. We also note that $\bar{\omega}$ being spatially one-dimensional allows the use of Agmon's inequality (2.2.5).

With $\bar{\omega}$ and $\tilde{\omega}$ thus defined, we state the following result by Al-Jaboori and Wirosoetisno [7] (using our definition of $\mathcal{G}_m$), which we will use frequently. Recall that

$\nu_0 = \mu\kappa_0^2$.

Theorem 11. *Assume that the initial data* $\boldsymbol{v}(0) \in L^2(\mathbb{T}^2)$ *and that* $|\Delta f|_{L^2(\mathbb{T}^2)} < \infty$. *Then there exists a time* $\mathcal{T}_0(|\boldsymbol{v}(0)|_{L^2(\mathbb{T}^2)})$ *and a constant* $c_5(\nu_0)$ *such that*

$$|\tilde{\omega}(t)|^2_{L^2(\mathbb{T}^2)} + \mu \int_t^{t+1} |\nabla\tilde{\omega}(\tau)|^2_{L^2(\mathbb{T}^2)}\,\mathrm{d}\tau \leq \varepsilon M_0/\kappa_0^2, \tag{3.2.9}$$

$$|\tilde{\omega}(t)|^2_{L^2(\mathbb{T}^2)} + \mu \int_0^{t} |\nabla\tilde{\omega}(\tau)|^2_{L^2(\mathbb{T}^2)} \mathrm{e}^{\nu_0(\tau - t)}\,\mathrm{d}\tau \leq \varepsilon M_0/\kappa_0^2 \tag{3.2.10}$$

for all $t \geq \mathcal{T}_0$, *where*

$$M_0 = c_5\, \mathcal{G}_2 \mathcal{G}_3 (1 + \mathcal{G}_0^2). \tag{3.2.11}$$

We note that the constants in [7] may include lengths, while ours do not, which accounts for the extra factor of κ_0^{-2}. We also note that our M_0 is a slight improvement on that given in [7], with a $\mathcal{G}_2\,\mathcal{G}_3$ dependence instead of $\mathcal{G}_3^2$. This is because the “worst” term in the original result (in the sense of requiring the highest order Grashof number to bound it) was $|\Delta f|_{L^2}|\Delta\omega|_{L^2}$, which was further bounded as $|\Delta f|_{L^2}|\Delta\omega|_{L^2} \leq c\,(|\Delta f|^2_{L^2} + |\Delta\omega|^2_{L^2})$ (by Young) in order to simplify a long sum. Thus the $|\Delta f|^2_{L^2}$ term could only be bounded by $\mathcal{G}_3^2$, but leaving it simply as $|\Delta f|_{L^2}|\Delta\omega|_{L^2}$ automatically gives us a $\mathcal{G}_2\,\mathcal{G}_3$ bound instead.

3.3 Consequences of different forms of forcing

Finally, before stating our main result, we consider different types of zonal forcing $\bar{f}$ and their effects on the flow. We give three different examples below, which are used often in numerical simulations.

$$\text{Bandwidth-limited:} \quad \bar{f} = \mathsf{P}_{\kappa_f} \bar{f} \qquad (\kappa_f \geq \kappa_0), \tag{3.3.1}$$

$$\text{Algebraic decay:} \quad |\bar{f}_{(0,k)}| \leq \frac{\nu_0^2\, \kappa_0^{s-1}\, |k|^{-s}}{\sqrt{2}\,\zeta(2+2s)^{1/2}}\, \mathcal{G}_0 \qquad (s > 5/2), \tag{3.3.2}$$

$$\text{Exponential decay:} \quad |\bar{f}_{(0,k)}| \leq \frac{\nu_0^2}{2\kappa_0} \left(\frac{2\gamma}{1+2\gamma}\right)^{1/2} \mathrm{e}^{\gamma(1-|k|/\kappa_0)}\, \mathcal{G}_0 \quad (\gamma > 0), \tag{3.3.3}$$

where $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. We note that the bandwidth-limited $\bar{f}$ is the conceptually important case to us; the algebraically and exponentially decaying cases can be considered as smoother and more “realistic”.

The requirement for (3.3.2) that $s > 5/2$ is purely to ensure $f \in H^2(\mathbb{T}^2)$, so that we can apply Theorem 11. The precise expressions for (3.3.2) and (3.3.3) have been chosen to guarantee that $|\nabla^{-1}\bar{f}|/(\mu\kappa_0)^2 \le \mathcal{G}_0$, in order to be consistent with the definition of the Grashof number given in (2.1.5):

$$|\bar{f}_{(0,k)}| \le \frac{\nu_0^2\,\kappa_0^{s-1}\,|k|^{-s}}{\sqrt{2}\zeta(2+2s)^{1/2}}\mathcal{G}_0 \qquad \text{implies that}$$

$$\begin{aligned}
|\nabla^{-1}\bar{f}|^2 &= \sum_{k_1=0} |f_{\boldsymbol{k}}|^2/|\boldsymbol{k}|^2 \le \sum_{k\in\langle 2\pi/L\rangle} \frac{\nu_0^4\,\kappa_0^{2s-2}\,\mathcal{G}_0^2}{2\zeta(2+2s)|k|^{2s+2}} = \frac{\nu_0^4\,\kappa_0^{2s-2}\,\mathcal{G}_0^2}{2\zeta(2+2s)} \sum_{k\in\langle 2\pi/L\rangle} |k|^{-2s-2} \\
&= \frac{\nu_0^4\,\kappa_0^{2s-2}\,\mathcal{G}_0^2}{\zeta(2+2s)} \sum_{k\ge\kappa_0} k^{-2s-2} \qquad\qquad \text{by symmetry} \\
&\le \frac{\nu_0^4\,\kappa_0^{2s-3}\,\mathcal{G}_0^2}{\zeta(2+2s)} \int_{\kappa_0}^{\infty} \frac{\mathrm{d}k}{k^{2s+2}} = \frac{\mu^4\,\kappa_0^4}{(2s+1)\zeta(2+2s)}\,\mathcal{G}_0^2,
\end{aligned}$$

which leads to the bound

$$|\nabla^{-1}\bar{f}|/(\mu\kappa_0)^2 \le \Big((2s+1)\zeta(2+2s)\Big)^{-1/2}\mathcal{G}_0 \le (6\,\zeta(2+2s))^{-1/2}\,\mathcal{G}_0 \le \mathcal{G}_0,$$

since ζ is decreasing in s and $\lim_{s\to\infty}\zeta(s) = 1$.

Considering $\bar{f}$ satisfying (3.3.3) instead,

$$|\bar{f}_{(0,k)}| \le \frac{\nu_0^2}{2\kappa_0}\left(\frac{2\gamma}{1+2\gamma}\right)^{1/2} \mathrm{e}^{\gamma(1-|k|/\kappa_0)}\,\mathcal{G}_0 \qquad \text{implies that}$$

$$\begin{aligned}
|\nabla^{-1}\bar{f}|^2 &\le \sum_{k_1=0} |f_{\boldsymbol{k}}|^2/|\boldsymbol{k}|^2 \le \frac{\nu_0^4\,\mathcal{G}_0^2}{4\,\kappa_0^2}\left(\frac{2\gamma}{1+2\gamma}\right) \sum_{k\in\langle 2\pi/L\rangle} \frac{\mathrm{e}^{2\gamma(1-|k|/\kappa_0)}}{k^2} \\
&= \frac{\nu_0^4\,\mathcal{G}_0^2}{\kappa_0^2}\left(\frac{\gamma}{1+2\gamma}\right) \sum_{k\ge\kappa_0} \frac{\mathrm{e}^{2\gamma(1-|k|/\kappa_0)}}{k^2} \qquad\qquad \text{by symmetry} \\
&\le \mu^4\kappa_0^4\,\mathcal{G}_0^2\left(\frac{\gamma}{1+2\gamma}\right)\mathrm{e}^{2\gamma} \sum_{k\ge\kappa_0} \mathrm{e}^{-2\gamma k/\kappa_0} \qquad\qquad \text{since } k \ge \kappa_0 \\
&= \mu^4\kappa_0^4\,\mathcal{G}_0^2\left(\frac{\gamma}{1+2\gamma}\right)(1 + \mathrm{e}^{-2\gamma} + \mathrm{e}^{-4\gamma} + \cdots) \\
&= \mu^4\kappa_0^4\,\mathcal{G}_0^2\left(\frac{\gamma}{1+2\gamma}\right)/(1-\mathrm{e}^{-2\gamma}) \le \mu^4\kappa_0^4\,\mathcal{G}_0^2 \qquad\qquad \text{for all } \gamma > 0,
\end{aligned}$$

leading to

$$|\nabla^{-1}\bar{f}|/(\mu\kappa_0)^2 \le \mathcal{G}_0.$$

With these forms of $\bar{f}$ in mind, we state and prove the following intermediate results.

Lemma 12. *Suppose ω satisfies* (3.1.24) *and define $\bar{\omega}^{<f} := \mathsf{P}_{\kappa_f}\bar{\omega}$, $\bar{\omega}^{>f} := \bar{\omega} - \bar{\omega}^{<f}$ for some $\kappa_f \ge \kappa_0$. Assume $\nu_0 = \mu\kappa_0^2 < 1$. Then there exists an absolute constant c_* such that:*

(a) if $\bar{f}$ satisfies (3.3.1), *then*

$$\int_0^t |\nabla\bar{\omega}^{>f}|^2_{L^2(\mathbb{T}^2)} \mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau \le 3\,c_*(\varepsilon M_0)^2/\nu_0^3\,; \tag{3.3.4}$$

(b) if $\bar{f}$ satisfies (3.3.2), *then*

$$\int_0^t |\nabla\bar{\omega}^{>f}|^2_{L^2(\mathbb{T}^2)} \mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau \le \quad c_*(\varepsilon M_0)^2/\nu_0^3 + \frac{8\nu_0}{(2s+1)\zeta(2s+2)}\left(\frac{\kappa_0}{\kappa_f}\right)^{2s+1}\mathcal{G}_0^2, \quad \textit{and} \tag{3.3.5}$$

(c) if $\bar{f}$ satisfies (3.3.3),

$$\int_0^t |\nabla\bar{\omega}^{>f}|^2_{L^2(\mathbb{T}^2)} \mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau \le \quad c_*(\varepsilon M_0)^2/\nu_0^3 + 8\,\nu_0\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\,\mathcal{G}_0^2. \tag{3.3.6}$$

Proof. We begin by multiplying (3.1.26) by $\bar{\omega}^{>f}$ in L^2:

$$(\partial_t\omega, \bar{\omega}^{>f}) + (\partial(\psi,\omega), \bar{\omega}^{>f}) + \frac{\kappa_0}{\varepsilon}(\partial_x\psi, \bar{\omega}^{>f}) = \mu(\Delta\omega, \bar{\omega}^{>f}) + (f, \bar{\omega}^{>f}). \tag{3.3.7}$$

The first term becomes

$$\begin{aligned}(\partial_t\omega, \bar{\omega}^{>f}) &= (\partial_t\tilde{\omega}, \bar{\omega}^{>f}) + (\partial_t\bar{\omega}^{<f}, \bar{\omega}^{>f}) + (\partial_t\bar{\omega}^{>f}, \bar{\omega}^{>f})\\ &= (\partial_t\bar{\omega}^{>f}, \bar{\omega}^{>f}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\omega}^{>f}|^2,\end{aligned} \tag{3.3.8}$$

by the orthogonality of $\bar{\omega}$ and $\tilde{\omega}$ (3.2.7). By splitting $\omega = \bar{\omega} + \tilde{\omega}$ and $\psi = \bar{\psi} + \tilde{\psi}$, the second term of (3.3.7) becomes

$$\begin{aligned}(\partial(\psi,\omega), \bar{\omega}^{>f}) &= (\partial(\psi,\bar{\omega}), \bar{\omega}^{>f}) + (\partial(\psi,\tilde{\omega}), \bar{\omega}^{>f})\\ &= (\partial(\bar{\omega}, \bar{\omega}^{>f}), \psi) + (\partial(\psi,\tilde{\omega}), \bar{\omega}^{>f}) && \text{by (3.1.18)}\\ &= (\partial(\psi,\tilde{\omega}), \bar{\omega}^{>f}) && \text{by (3.2.8)}\end{aligned}$$

$$
\begin{aligned}
&= (\partial(\bar\psi,\tilde\omega),\bar\omega^{>f}) + (\partial(\tilde\psi,\tilde\omega),\bar\omega^{>f}) \\
&= (\partial(\bar\omega^{>f},\bar\psi),\tilde\omega) + (\partial(\tilde\psi,\tilde\omega),\bar\omega^{>f}) && \text{by (3.1.18)} \\
&= (\partial(\tilde\psi,\tilde\omega),\bar\omega^{>f}) && \text{by (3.2.8).} \qquad (3.3.9)
\end{aligned}
$$

Integration by parts shows that the third term of (3.3.7) is 0:

$$
\begin{aligned}
\frac{\kappa_0}{\varepsilon}(\partial_x\psi,\bar\omega^{>f}) &= \frac{\kappa_0}{\varepsilon}(\partial_x\tilde\psi,\bar\omega^{>f}) + \frac{\kappa_0}{\varepsilon}(\partial_x\bar\psi,\bar\omega^{>f}) \\
&= \frac{\kappa_0}{\varepsilon}(\partial_x\tilde\psi,\bar\omega^{>f}) \quad \text{since } \partial_x\bar\psi = 0 \\
&= \frac{\kappa_0}{\varepsilon}\sum_{|\boldsymbol{k}|>\kappa_f} k_1\tilde\psi_{\boldsymbol{k}}\,\overline{\bar\omega_{\boldsymbol{k}}} = \frac{\kappa_0}{\varepsilon}\Bigg(\sum_{\substack{|\boldsymbol{k}|>\kappa_f\\ k_1=0}} k_1\tilde\psi_{\boldsymbol{k}}\,\overline{\bar\omega_{\boldsymbol{k}}} + \sum_{\substack{|\boldsymbol{k}|>\kappa_f\\ k_1\neq 0}} k_1\tilde\psi_{\boldsymbol{k}}\,\overline{\bar\omega_{\boldsymbol{k}}}\Bigg) \\
&= 0. && (3.3.10)
\end{aligned}
$$

We note the following property of the Laplacian:

$$
\begin{aligned}
(-\Delta u, u) &= -\int_{\mathbb{T}^2} u\,(\partial^2_{xx}u + \partial^2_{yy}u)\,\mathrm{d}\boldsymbol{x} \\
&= -\int_{\mathbb{T}^2} u\,\partial^2_{xx}u\,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{T}^2} u\,\partial^2_{yy}u\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{T}^2}\Big((\partial_x u)^2 + (\partial_y u)^2\Big)\,\mathrm{d}\boldsymbol{x} && \text{by integration by parts} \\
&= |\nabla u|^2 && (3.3.11)
\end{aligned}
$$

for $u \in H^2(\mathbb{T}^2)$, by which the first term on the right hand side of (3.3.7) becomes

$$
\begin{aligned}
\mu(\Delta\omega,\bar\omega^{>f}) &= \mu(\Delta\bar\omega,\bar\omega^{>f}) + \mu(\Delta\tilde\omega,\bar\omega^{>f}) \\
&= \mu(\Delta\bar\omega,\bar\omega^{>f}) && \text{by (3.2.7)} \\
&= \mu(\Delta\bar\omega^{<f},\bar\omega^{>f}) + \mu(\Delta\bar\omega^{>f},\bar\omega^{>f}) \\
&= \mu(\Delta\bar\omega^{>f},\bar\omega^{>f}) = -\mu|\nabla\bar\omega^{>f}|^2. && (3.3.12)
\end{aligned}
$$

Collecting (3.3.8) to (3.3.12) gives

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar\omega^{>f}|^2 + \mu|\nabla\bar\omega^{>f}|^2 = -(\partial(\tilde\psi,\tilde\omega),\bar\omega^{>f}) + (f,\bar\omega^{>f}). \qquad (3.3.13)
$$

The first term on the right hand side of this can be bounded by

$$\begin{aligned}
|(\partial(\tilde{\psi},\tilde{\omega}),\bar{\omega}^{>f})| &= |(\partial(\bar{\omega}^{>f},\tilde{\psi}),\tilde{\omega})| && \text{by (3.1.18)}\\
&\le |\nabla\tilde{\psi}|_\infty|\tilde{\omega}|_2|\nabla\bar{\omega}^{>f}|_2 && \text{by Hölder}\\
&\le \frac{2}{\mu}|\nabla\tilde{\psi}|_\infty^2|\tilde{\omega}|^2+\frac{\mu}{8}|\nabla\bar{\omega}^{>f}|^2 && \text{by Young.}
\end{aligned}\tag{3.3.14}$$

Similarly, we bound the forcing term by

$$\begin{aligned}
(f,\bar{\omega}^{>f}) = (\bar{f}^{>f},\bar{\omega}^{>f}) &\le |\nabla^{-1}\bar{f}^{>f}|_2|\nabla\bar{\omega}^{>f}|_2\\
&\le \frac{2}{\mu}|\nabla^{-1}\bar{f}^{>f}|^2+\frac{\mu}{8}|\nabla\bar{\omega}^{>f}|^2.
\end{aligned}\tag{3.3.15}$$

Thus (3.3.13) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\omega}^{>f}|^2+\frac{3}{2}\mu|\nabla\bar{\omega}^{>f}|^2 \le \frac{4}{\mu}|\nabla\tilde{\psi}|_\infty^2|\tilde{\omega}|^2+\frac{4}{\mu}|\nabla^{-1}\bar{f}^{>f}|^2.$$

Assuming large enough t, we can use the bound on $|\tilde{\omega}|$ from Theorem 11 to give

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\omega}^{>f}|^2+\frac{3}{2}\,\mu|\nabla\bar{\omega}^{>f}|^2 \le c\,\frac{\varepsilon M_0}{\nu_0}|\nabla\tilde{\psi}|_\infty^2+\frac{4}{\mu}|\nabla^{-1}\bar{f}^{>f}|^2.$$

Using Agmon's inequality (2.2.6) on the right hand side leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\omega}^{>f}|^2+\frac{3}{2}\mu|\nabla\bar{\omega}^{>f}|^2 \le \frac{c\,\varepsilon M_0}{\nu_0\kappa_0^2}|\nabla\tilde{\omega}|^2+\frac{4}{\mu}|\nabla^{-1}\bar{f}^{>f}|^2.$$

We then apply Poincaré's inequality (3.1.32) on the $|\nabla\bar{\omega}^{>f}|$ term on the left hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\omega}^{>f}|^2+\mu\,|\bar{\omega}^{>f}|^2+\frac{\mu}{2}|\nabla\bar{\omega}^{>f}|^2 \le \frac{c\,\varepsilon M_0}{\nu_0\kappa_0^2}|\nabla\tilde{\omega}|^2+\frac{4}{\mu}|\nabla^{-1}\bar{f}^{>f}|^2$$

and multiply by $\mathrm{e}^{\nu_0 t}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu_0 t}|\bar{\omega}^{>f}|^2)+\frac{\mu}{2}\,\mathrm{e}^{\nu_0 t}|\nabla\bar{\omega}^{>f}|^2 \le \frac{c\,\varepsilon M_0}{\nu_0\kappa_0^2}|\nabla\tilde{\omega}|^2\mathrm{e}^{\nu_0 t}+\frac{4}{\mu}\,\mathrm{e}^{\nu_0 t}|\nabla^{-1}\bar{f}^{>f}|^2,$$

then integrate in time and multiply by $\mathrm{e}^{-\nu_0 t}$ to obtain

$$\begin{aligned}
&|\bar{\omega}^{>f}(t)|^2+\frac{\mu}{2}\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau\\
&\qquad\le \mathrm{e}^{-\nu_0 t}|\bar{\omega}^{>f}(0)|^2+\frac{c\,\varepsilon M_0}{\nu_0\kappa_0^2}\int_0^t|\nabla\tilde{\omega}|^2\mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau+\frac{4}{\mu\nu_0}|\nabla^{-1}\bar{f}^{>f}|^2
\end{aligned}$$

$$\le \frac{c_*(\varepsilon M_0)^2}{2\,\nu_0^2\kappa_0^2} + \frac{4}{\mu\nu_0}|\nabla^{-1}\bar{f}^{>f}|^2, \tag{3.3.16}$$

where we have used (3.2.10) and assumed t is large enough to absorb the $|\bar{\omega}^{>f}(0)|^2$ term into the $(\varepsilon M_0)^2$ term, in combination with the adjusted constant c_*.

We now consider the consequences of the hypotheses (3.3.1) to (3.3.3). When $\bar{f}$ satisfies (3.3.1), we have $\bar{f}^{>f} = 0$ by definition, so that (3.3.16) becomes

$$\frac{\mu}{2}\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le \frac{c_*(\varepsilon M_0)^2}{2\,\nu_0^2\kappa_0^2}, \tag{3.3.17}$$

where we have used Lemma 9 and dropped the first term on the left hand side. With our assumption that $\nu_0 < 1$ and hence $\mathrm{e}^{\nu_0} < 3$, we arrive at

$$\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le 3\,c_*(\varepsilon M_0)^2/\nu_0^3,$$

which is (3.3.4).

When $\bar{f}$ satisfies (3.3.2), we have

$$\begin{aligned}
|\nabla^{-1}\bar{f}^{>f}|^2 &= \sum_{\substack{|\boldsymbol{k}|>\kappa_f\\ k_1=0}} |f_{\boldsymbol{k}}|^2/|\boldsymbol{k}|^2 \le \sum_{\substack{|\boldsymbol{k}|>\kappa_f\\ k_1=0}} \frac{\nu_0^4\,\kappa_0^{2s-2}\,\mathcal{G}_0^2}{2\zeta(2+2s)}|\boldsymbol{k}|^{-2s-2}\\
&= \frac{\nu_0^4\,\kappa_0^{2s-2}\,\mathcal{G}_0^2}{2\zeta(2+2s)}\sum_{\substack{|\boldsymbol{k}|>\kappa_f\\ k_1=0}} |\boldsymbol{k}|^{-2s-2}\\
&= \frac{\nu_0^4\,\kappa_0^{2s-2}\,\mathcal{G}_0^2}{\zeta(2+2s)}\sum_{k>\kappa_f} k^{-2s-2} \qquad \text{by symmetry}\\
&\le \frac{\nu_0^4\,\kappa_0^{2s-3}\,\mathcal{G}_0^2}{\zeta(2+2s)}\int_{\kappa_f}^\infty \frac{\mathrm{d}k}{k^{2s+2}} = \frac{\nu_0^4\,(\kappa_0/\kappa_f)^{2s+1}\,\mathcal{G}_0^2}{(2s+1)\zeta(2+2s)\kappa_0^4},
\end{aligned} \tag{3.3.18}$$

so (3.3.16) becomes

$$\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le c_*\,(\varepsilon M_0)^2/\nu_0^3 + \frac{8(\kappa_0/\kappa_f)^{2s+1}\nu_0}{(2s+1)\zeta(2s+2)}\mathcal{G}_0^2, \tag{3.3.19}$$

which is (3.3.5).

Finally, when $\bar{f}$ satisfies (3.3.3),

$$|\nabla^{-1}\bar{f}^{>f}|^2 = \sum_{\substack{|\boldsymbol{k}|>\kappa_f\\ k_1=0}} |f_{\boldsymbol{k}}|^2/|\boldsymbol{k}|^2 \le \sum_{\substack{|\boldsymbol{k}|>\kappa_f\\ k_1=0}} \frac{\nu_0^4\,\mathcal{G}_0^2}{4\,\kappa_0^2}\Big(\frac{2\gamma}{1+2\gamma}\Big)\frac{\mathrm{e}^{2\gamma(1-|\boldsymbol{k}|/\kappa_0)}}{|\boldsymbol{k}|^2}$$

$$
\begin{aligned}
&= \sum_{k>\kappa_f} \frac{\nu_0^4\,\mathcal{G}_0^2}{\kappa_0^2}\Big(\frac{\gamma}{1+2\gamma}\Big)\frac{\mathrm{e}^{2\gamma(1-k/\kappa_0)}}{k^2} && \text{by symmetry}\\
&\le \frac{\nu_0^4\,\mathcal{G}_0^2}{\kappa_0^4}\Big(\frac{\gamma}{1+2\gamma}\Big)\sum_{k>\kappa_f}\mathrm{e}^{2\gamma(1-k/\kappa_0)} && \text{since } k>\kappa_f\ge\kappa_0\\
&\le \frac{\nu_0^4\,\mathcal{G}_0^2}{\kappa_0^4}\Big(\frac{\gamma}{1+2\gamma}\Big)\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}(1+\mathrm{e}^{-2\gamma}+\mathrm{e}^{-4\gamma}+\cdots)\\
&= \frac{\nu_0^4\,\mathcal{G}_0^2}{\kappa_0^4}\Big(\frac{\gamma}{1+2\gamma}\Big)\frac{\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}}{1-\mathrm{e}^{-2\gamma}} \le \frac{\nu_0^4}{\kappa_0^4}\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\,\mathcal{G}_0^2, && (3.3.20)
\end{aligned}
$$

so (3.3.16) becomes

$$
\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le \frac{c_*(\varepsilon M_0)^2}{\nu_0^3} + 8\,\nu_0\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\mathcal{G}_0^2, \tag{3.3.21}
$$

giving (3.3.6). □

3.4 Bounds on the number of determining modes

We now have all the definitions and intermediate results required for the proof of our main result, which we state below. We assume, as with Lemma 12, that $\nu_0 < 1$.

Theorem 13 (Determining modes for the β-plane)**.** *Let $\delta\omega$ be the solution of* (3.1.28) *with $f \in H^2(\mathbb{T}^2)$. Then the low modes are determining, i.e.* $\lim_{t\to\infty}|\mathsf{P}_\kappa\delta\omega(t)|_{L^2(\mathbb{T}^2)} = 0$ *implies that* $\lim_{t\to\infty}|\delta\omega(t)|_{L^2(\mathbb{T}^2)} = 0$*, if any of the following hold for constants c_6, c_7, c_8 and ε sufficiently small:*

(a) if $\bar{f}$ satisfies (3.3.1) *and*

$$
\kappa/\kappa_0 > c_6\,\max\{(\varepsilon M_0)^{1/4}, (\kappa_f/\kappa_0)^{3/8}\,\mathcal{G}_0^{1/4}\}; \tag{3.4.1}
$$

(b) if $\bar{f}$ satisfies (3.3.2) *and*

$$
\kappa/\kappa_0 > c_7\,\max\{(\varepsilon M_0)^{1/4}, \mathcal{G}_0^{(2s+5)/(8s+14)}\}, \quad \textit{or} \tag{3.4.2}
$$

(c) if $\bar{f}$ satisfies (3.3.3) *and*

$$
\kappa/\kappa_0 > c_8\,\max\{(\varepsilon M_0)^{1/4}, F_\gamma(\nu_0^{-1/2}\,\mathcal{G}_0)^{3/8}\,\mathcal{G}_0^{1/4}\}, \tag{3.4.3}
$$

where the function F_γ is defined in (3.4.40) *below.*

The smallness requirement on ε is given in (3.4.28), (3.4.35) and (3.4.41) below for $\bar{f}$ satisfying (3.3.1), (3.3.2) and (3.3.3) respectively. Technically these requirements are not essential and can be removed in exchange for adding another ε-dependent term in the bounds (3.4.1), (3.4.2) and (3.4.3); we have chosen to include them purely to simplify the statement of the theorem.

We note also that for large u, $F_\gamma(u) \approx \log u/(2\gamma)$, so that the last term in (3.4.3) scales (up to a logarithm) as $\mathcal{G}_0^{1/4}$.

Our proof below suggests that the $(\varepsilon M_0)^{1/4}$ bounds can be thought of as an effect of the non-zonal $\tilde{f}$ and the $\mathcal{G}_0$ bounds as arising from the zonal $\bar{f}$. For small ε, one may therefore consider the rotating Navier–Stokes equations as a combination of a one-dimensional (zonal) "average" and a two-dimensional small (non-zonal) noise, which agrees with the physical expectations [6]. It is therefore unlikely that, whilst using our same approach, a bound with a smaller power of $\mathcal{G}_0$ could be obtained.

Finally it is worth noting that since we require $s > 5/2$ in order to apply Lemma 12, the worst case dependence we have is of the order $\mathcal{G}_0^{5/17}$.

Proof. We begin by multiplying (3.1.28) by $\delta\omega^>$ in L^2 to obtain

$$(\partial_t \delta\omega, \delta\omega^>) + (\partial(\psi^\sharp, \delta\omega), \delta\omega^>) + (\partial(\delta\psi, \omega), \delta\omega^>) + \frac{\kappa_0}{\varepsilon}(\partial_x \delta\psi, \delta\omega^>) = (\mu\Delta\delta\omega, \delta\omega^>). \tag{3.4.4}$$

Fourier expansion shows that the κ_0/ε term is 0:

$$\begin{aligned}\frac{\kappa_0}{\varepsilon}(\partial_x \delta\psi, \delta\omega^>) &= \frac{\kappa_0}{\varepsilon}\sum_{|\boldsymbol{k}|>\kappa} \mathrm{i}k_1 \delta\psi_{\boldsymbol{k}}\, \overline{\delta\omega_{\boldsymbol{k}}} \\ &= -\frac{\kappa_0}{\varepsilon}\sum_{|\boldsymbol{k}|>\kappa} -\mathrm{i}k_1 |\boldsymbol{k}|^2 \delta\psi_{\boldsymbol{k}} \overline{\delta\psi_{\boldsymbol{k}}} \\ &= 0 \qquad \text{by symmetry,}\end{aligned} \tag{3.4.5}$$

so that (3.4.4) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega^>|^2 + \mu|\nabla\delta\omega^>|^2 = -(\partial(\psi^\sharp, \delta\omega), \delta\omega^>) - (\partial(\delta\psi^<, \omega), \delta\omega^>) - (\partial(\delta\psi^>, \omega), \delta\omega^>). \tag{3.4.6}$$

For the first term on the right hand side, (3.1.17) implies that $(\partial(\psi^\sharp, \delta\omega^>), \delta\omega^>) = 0$, so

$$(\partial(\psi^\sharp, \delta\omega), \delta\omega^>) = (\partial(\psi^\sharp, \delta\omega^<), \delta\omega^>). \tag{3.4.7}$$

We split $\omega = \bar\omega + \tilde\omega$ to write the last term of (3.4.6) as

$$(\partial(\delta\psi^>, \omega), \delta\omega^>) = (\partial(\delta\psi^>, \bar\omega), \delta\omega^>) + (\partial(\delta\psi^>, \tilde\omega), \delta\omega^>). \tag{3.4.8}$$

Using (3.2.8), the first term on the right hand side of this becomes

$$(\partial(\delta\psi^>, \bar\omega), \delta\omega^>) = (\partial(\delta\tilde\psi^>, \bar\omega), \delta\tilde\omega^>). \tag{3.4.9}$$

In order to apply the bounds we obtained in Lemma 12, we write $\bar\omega = \bar\omega^{<f} + \bar\omega^{>f}$, where $\bar\omega^{<f} = \mathsf{P}_{\kappa_f}\bar\omega$ and $\bar\omega^{>f} = \bar\omega - \bar\omega^{<f}$. Now (3.4.9) becomes

$$(\partial(\delta\tilde\psi^>, \bar\omega), \delta\tilde\omega^>) = (\partial(\delta\tilde\psi^>, \bar\omega^{<f}), \delta\tilde\omega^>) + (\partial(\delta\tilde\psi^>, \bar\omega^{>f}), \delta\tilde\omega^>). \tag{3.4.10}$$

We thus expand (3.4.6) as

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega^>|^2 + \mu|\nabla\delta\omega^>|^2 \\
&\quad = -(\partial(\psi^\sharp, \delta\omega^<), \delta\omega^>) - (\partial(\delta\psi^<, \omega), \delta\omega^>) - (\partial(\delta\psi^>, \tilde\omega), \delta\omega^>) \\
&\qquad - (\partial(\delta\tilde\psi^>, \bar\omega^{<f}), \delta\tilde\omega^>) - (\partial(\delta\tilde\psi^>, \bar\omega^{>f}), \delta\tilde\omega^>).
\end{aligned} \tag{3.4.11}$$

We bound the first two terms on the right hand side by

$$\begin{aligned}
|(\partial(\psi^\sharp, \delta\omega^<), \delta\omega^>)| &= |(\partial(\delta\omega^>, \psi^\sharp), \delta\omega^<)| && \text{by (3.1.18)} \\
&\le |\nabla\psi^\sharp|_\infty |\nabla\delta\omega^>|_2 |\delta\omega^<|_2 && \text{by Hölder} \\
&\le \frac{4}{\mu}|\nabla\psi^\sharp|_\infty^2 |\delta\omega^<|_2^2 + \frac{\mu}{16}|\nabla\delta\omega^>|_2^2 && \text{by Young,}
\end{aligned} \tag{3.4.12}$$

and

$$\begin{aligned}
|(\partial(\delta\psi^<, \omega), \delta\omega^>)| &= |(\partial(\delta\omega^>, \delta\psi^<), \omega)| \\
&\le |\nabla\delta\omega^>|_2 |\nabla\delta\psi^<|_\infty |\omega|_2 \\
&\le \frac{4}{\mu}|\nabla\delta\psi^<|_\infty^2 |\omega|_2^2 + \frac{\mu}{16}|\nabla\delta\omega^>|_2^2.
\end{aligned} \tag{3.4.13}$$

The third term on the right hand side of (3.4.11) can be bounded as

$$
\begin{aligned}
|(\partial(\delta\psi^>,\tilde\omega),\delta\omega^>)| &\le |\nabla\delta\psi^>|_\infty|\nabla\tilde\omega|_2|\delta\omega^>|_2 \\
&\le c\,|\nabla\delta\psi^>|^{1/2}|\nabla\delta\omega^>|^{1/2}|\delta\omega^>||\nabla\tilde\omega| && \text{by (2.2.6)} \\
&\le \frac{c}{\kappa}|\nabla\delta\omega^>||\delta\omega^>||\nabla\tilde\omega| && \text{by (3.1.32)} \\
&\le \frac{\mu}{16}|\nabla\delta\omega^>|^2+\frac{c}{\mu\kappa^2}|\nabla\tilde\omega|^2|\delta\omega^>|^2, && (3.4.14)
\end{aligned}
$$

and the fourth term as

$$
\begin{aligned}
|(\partial(\delta\tilde\psi^>,\bar\omega^{<f}),\delta\tilde\omega^>)| &= |(\partial(\delta\tilde\psi^>,\nabla\bar\omega^{<f}),\nabla\delta\tilde\psi^>)| && \text{by integration by parts} \\
&\le c\,|\Delta\bar\omega^{<f}|_\infty|\nabla\delta\psi^>|_2^2 \\
&\le c\,\frac{\kappa_0^{1/2}}{\kappa^2}|\Delta\bar\omega^{<f}|^{1/2}|\nabla^3\bar\omega^{<f}|^{1/2}|\delta\omega^>|^2 && \text{by (3.1.32) and (2.2.5)} \\
&\le c\,\frac{(\kappa_0\kappa_f^3)^{1/2}}{\kappa^3}|\nabla\bar\omega^{<f}|\,|\delta\omega^>|\,|\nabla\delta\omega^>| && \text{by (3.1.33)} \\
&\le \frac{\mu}{16}|\nabla\delta\omega^>|^2+c\,\frac{\kappa_0\kappa_f^3}{\mu\kappa^6}|\nabla\omega|^2|\delta\omega^>|^2. && (3.4.15)
\end{aligned}
$$

We bound the final term of (3.4.11) by

$$
\begin{aligned}
|(\partial(\delta\psi^>,\bar\omega^{>f}),\delta\omega^>)| &= |(\partial(\delta\omega^>,\delta\psi^>),\bar\omega^{>f})| \\
&\le |\nabla\delta\omega^>|_2|\nabla\delta\psi^>|_2|\bar\omega^{>f}|_\infty \\
&\le c\,\kappa_0^{1/2}|\bar\omega^{>f}|^{1/2}|\nabla\bar\omega^{>f}|^{1/2}|\nabla\delta\omega^>||\nabla\delta\psi^>| && \text{by (2.2.5)} \\
&\le c\left(\frac{\kappa_0}{\kappa_f}\right)^{1/2}|\nabla\bar\omega^{>f}||\nabla\delta\omega^>||\nabla\delta\psi^>| && \text{by (3.1.32)} \\
&\le \frac{c}{\kappa}\left(\frac{\kappa_0}{\kappa_f}\right)^{1/2}|\nabla\bar\omega^{>f}|\,|\nabla\delta\omega^>|\,|\delta\omega^>| && \text{by (3.1.32)} \\
&\le \frac{\mu}{16}|\nabla\delta\omega^>|^2+\frac{c}{\mu\kappa^2}\frac{\kappa_0}{\kappa_f}|\nabla\bar\omega^{>f}|^2|\delta\omega^>|^2 && \text{by Young.} \quad (3.4.16)
\end{aligned}
$$

Collating these and rearranging, we arrive at

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega^>|^2+\mu\,|\nabla\delta\omega^>|^2 \le \frac{8}{\mu}|\nabla\psi^\sharp|_\infty^2|\delta\omega^<|^2+\frac{8}{\mu}|\nabla\delta\psi^<|_\infty^2|\omega|^2+\frac{c}{\mu\kappa^2}|\nabla\tilde\omega|^2|\delta\omega^>|^2 \\
+c\,\frac{\kappa_0\kappa_f^3}{\mu\kappa^6}|\nabla\omega|^2|\delta\omega^>|^2+\frac{c\,\kappa_0}{\mu\kappa^2\kappa_f}|\nabla\bar\omega^{>f}|^2|\delta\omega^>|^2.
\end{aligned}
$$

Applying (3.1.32) on the $|\nabla\delta\omega^{>}|^2$ term on the left hand side and rearranging gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega^{>}|^2 + |\delta\omega^{>}|^2\Big(\mu\kappa^2 - \frac{c}{\mu\kappa^2}|\nabla\tilde{\omega}|^2 - c\,\frac{\kappa_0\kappa_f^3}{\mu\kappa^6}|\nabla\omega|^2 - \frac{c\,\kappa_0}{\mu\kappa^2\kappa_f}|\nabla\bar{\omega}^{>f}|^2\Big) \\ &\qquad \le \frac{8}{\mu}|\nabla\psi^{\sharp}|_\infty^2|\delta\omega^{<}|^2 + \frac{8}{\mu}|\nabla\delta\psi^{<}|_\infty^2|\omega|^2. \end{aligned} \tag{3.4.17}$$

We now apply Lemma 8, with

$$\begin{aligned} \rho &= \mu\kappa^2 - \frac{c}{\mu\kappa^2}|\nabla\tilde{\omega}|^2 - \frac{c\,\kappa_0\kappa_f^3}{\mu\kappa^6}|\nabla\omega|^2 - \frac{c\,\kappa_0}{\mu\kappa^2\kappa_f}|\nabla\bar{\omega}^{>f}|^2, \\ \sigma &= \frac{8}{\mu}(|\nabla\psi^{\sharp}|_\infty^2|\delta\omega^{<}|^2 + |\nabla\delta\psi^{<}|_\infty^2|\omega|^2), \\ \xi &= |\delta\omega^{>}|^2, \end{aligned} \tag{3.4.18}$$

i.e. ρ is the bracket on the left hand side of (3.4.17) and σ is the right hand side. In order to validate that the hypothesis of the lemma concerning σ is met, we quote the following result from [7], which give bounds on the derivatives of the vorticity:

$$|\nabla^m\omega(t)|_{L^2(\mathbb{T}^2)}^2 + \mu\int_0^t |\nabla^{m+1}\omega|_{L^2(\mathbb{T}^2)}^2 \mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau \le c(m)\frac{\mathcal{G}_m^2(1+c'(m)\nu_0^2\mathcal{G}_0^2)^m}{(\mu\kappa_0)^{2m-2}} \tag{3.4.19}$$

for all $t \ge T_m(|\boldsymbol{v}(0)|, |\nabla^{m-1}f|;\mu)$; we note that the bounds themselves are independent of the initial data. Thus the hypothesis concerning σ is met because $|\nabla\omega|$ is bounded when integrated over time and $|\delta\omega^{<}(t)| \to 0$ as $t \to \infty$ by construction (since finite-dimensional norms are equivalent).

The hypothesis on ξ holds since the regularity of the 2D Navier–Stokes equations directly implies that $|\omega|$ has a continuous derivative. We therefore need the hypothesis on ρ to be fulfilled, which would follow from

$$\limsup_{t\to\infty}\int_t^{t+1}\Big(\frac{1}{\mu\kappa^2}|\nabla\tilde{\omega}|^2 + \frac{\kappa_0\kappa_f^3}{\mu\kappa^6}|\nabla\omega|^2 + \frac{\kappa_0}{\mu\kappa^2\kappa_f}|\nabla\bar{\omega}^{>f}|^2\Big)\,\mathrm{d}\tau < c\mu\kappa^2, \tag{3.4.20}$$

which in turn is implied by

$$\limsup_{t\to\infty}\int_t^{t+1}|\nabla\tilde{\omega}|^2\,\mathrm{d}\tau < c\,\mu^2\kappa^4, \tag{3.4.21}$$

$$\limsup_{t\to\infty}\int_t^{t+1}|\nabla\omega|^2\,\mathrm{d}\tau < \frac{c\mu^2\kappa^8}{\kappa_0\kappa_f^3}, \qquad \text{and} \tag{3.4.22}$$

$$\limsup_{t\to\infty}\int_t^{t+1}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau < c\,\mu^2\kappa^4\frac{\kappa_f}{\kappa_0}. \tag{3.4.23}$$

For the first of these conditions, we recall that (3.2.9) implies

$$\int_t^{t+1} |\nabla\tilde{\omega}|^2 \,\mathrm{d}\tau \le \varepsilon M_0/\nu_0,$$

so (3.4.21) follows when

$$\kappa/\kappa_0 > c\,(\varepsilon M_0/\nu_0^3)^{1/4}. \tag{3.4.24}$$

By (3.4.19), the second condition (3.4.22) is implied when

$$c\,\mathcal{G}_0^2\nu_0 < \mu^2\kappa^8/(\kappa_0\kappa_f^3) \iff \kappa/\kappa_0 > c\,\nu_0^{-1/8}(\kappa_f/\kappa_0)^{3/8}\mathcal{G}_0^{1/4}. \tag{3.4.25}$$

In order for (3.4.23) to be fulfilled, we first consider the case when $\bar{f}$ satisfies (3.3.1). Recalling (3.3.4), we apply Lemma 9, so that

$$\int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2 \,\mathrm{d}\tau \le c\,(\varepsilon M_0)^2/\nu_0^3, \tag{3.4.26}$$

i.e. (3.4.23) is met when

$$\kappa/\kappa_0 > c\,(\varepsilon M_0)^{1/2}\nu_0^{-5/4}(\kappa_0/\kappa_f)^{1/4}. \tag{3.4.27}$$

This bound is weaker than that of (3.4.24) when

$$\varepsilon M_0 \le c\,\nu_0^2(\kappa_f/\kappa_0), \tag{3.4.28}$$

which we will assume. Combining (3.4.24), (3.4.25) and (3.4.27), we arrive at (3.4.1).

When $\bar{f}$ instead satisfies (3.3.2), we apply Lemma 9 to (3.3.5) to obtain

$$\int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2 \,\mathrm{d}\tau \le c\,(\varepsilon M_0)^2/\nu_0^3 + c\,c_\zeta(s)\nu_0(\kappa_0/\kappa_f)^{2s+1}\,\mathcal{G}_0^2 =: I_1 \tag{3.4.29}$$

where $1/c_\zeta(s) := (2s+1)\zeta(2s+2)$. Thus (3.4.23) would be satisfied when $I_1 < c\,\mu^2\kappa^4(\kappa_f/\kappa_0)$. Analogously to what we did to (3.4.20), this is implied by

$$(\kappa/\kappa_0)^4 > c\,(\varepsilon M_0)^2\nu_0^{-5}(\kappa_0/\kappa_f), \qquad \text{and} \tag{3.4.30}$$

$$(\kappa/\kappa_0)^4 > c\,c_\zeta(s)\,\nu_0^{-1}(\kappa_0/\kappa_f)^{2s+2}\,\mathcal{G}_0^2. \tag{3.4.31}$$

Since both (3.4.22) and (3.4.31) must be met, we equate these bounds to find

$$(\kappa_f/\kappa_0)^{2s+7/2} = c\,c_\zeta(s)\nu_0^{-1/2}\,\mathcal{G}_0 \tag{3.4.32}$$

which fixes κ_f, turning both (3.4.22) and (3.4.31) to

$$\kappa/\kappa_0 > c\,(c_\zeta(s)^{3/2}\nu_0^{-(s+5/2)}\mathcal{G}_0^{2s+5})^{1/(8s+14)}. \tag{3.4.33}$$

Using κ_f determined in (3.4.32), (3.4.30) becomes

$$\kappa/\kappa_0 > c_s\,(\varepsilon M_0)^{1/2}\nu_0^{-5/4+1/(16s+28)}\mathcal{G}_0^{-1/(8s+14)} \tag{3.4.34}$$

where $c_s = c\,c_\zeta(s)^{-1/(8s+14)}$, noting that since we require $s > 5/2$, the exponent of $\mathcal{G}_0$ lies between $-1/34$ and 0, giving a weak dependence. This bound is dominated by that of (3.4.24) when

$$\varepsilon M_0 \le c\,c_s^{-4}\nu_0^{2-1/(4s+7)}\mathcal{G}_0^{2/(4s+7)}. \tag{3.4.35}$$

Assuming this, (3.4.2) follows from (3.4.24) and (3.4.33).

Finally, when $\bar{f}$ satisfies (3.3.3), Lemma 9 and (3.3.6) imply that

$$\int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le c\,(\varepsilon M_0)^2/\nu_0^3 + c\,\nu_0\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\,\mathcal{G}_0^2. \tag{3.4.36}$$

As in the previous case, (3.4.23) would be satisfied if the following both hold:

$$(\kappa/\kappa_0)^4 > c\,(\varepsilon M_0)^2\nu_0^{-5}(\kappa_0/\kappa_f), \qquad \text{and} \tag{3.4.37}$$

$$(\kappa/\kappa_0)^4 > c\,\nu_0^{-1}\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}(\kappa_0/\kappa_f)\,\mathcal{G}_0^2. \tag{3.4.38}$$

Equating (3.4.25) and (3.4.38) gives

$$(\kappa_f/\kappa_0)^{5/2}\,\mathrm{e}^{2\gamma(\kappa_f/\kappa_0-1)} = c_\gamma\,\nu_0^{-1/2}\,\mathcal{G}_0, \tag{3.4.39}$$

which can be inverted to give

$$\kappa_f/\kappa_0 = F_\gamma(\mathcal{G}_0/\nu_0^{1/2}) \quad \text{where} \quad F_\gamma^{-1}(y) = y^{5/2}\mathrm{e}^{2\gamma(y-1)}/c_\gamma. \tag{3.4.40}$$

With κ_f thus fixed, (3.4.24) would dominate (3.4.37) when

$$\varepsilon M_0 \le c\,\nu_0^2(\kappa_f/\kappa_0). \tag{3.4.41}$$

Assuming this, (3.4.3) follows from (3.4.24) and (3.4.25). $\square$

Chapter 4

Determining modes on the sphere

In this chapter, we state and prove our theorem concerning the number of determining modes on the sphere. We begin by introducing the necessary definitions and properties of functions over S^2 in Section 4.1, followed by deriving the spherical equivalent of Lemma 12 in Section 4.3. With these results established, we prove our theorem in Section 4.4.

One could argue that due to the more "realistic" nature of the domain, this chapter is more useful in practical applications. We also note, however, that despite the β-plane of the previous chapter being an approximation of the sphere, the results of the two chapters are of the same order, supporting our argument that the β-plane is a good approximation for our purposes.

4.1 Definitions and inequalities on the sphere

We begin by defining our coordinate system. The unit sphere is defined by $S^2 := \{(\theta, \phi, r = 1) : \theta \in [0, \pi], \phi \in [0, 2\pi)\}$, where θ is the polar angle or latitude (with $\theta = 0$ corresponding to the north pole) and ϕ is the azimuth angle or longitude. The corresponding Jacobian (determinant) is $\sin\theta$, so that the integral of a scalar function u over the sphere is

$$\int_{S^2} u \,\mathrm{d}A := \int_0^\pi \int_0^{2\pi} u(\theta,\phi)\sin\theta \,\mathrm{d}\phi\,\mathrm{d}\theta.$$

We recall that when there is no room for confusion, we use the notation $|\cdot| = |\cdot|_{L^2}$, $|\cdot|_p = |\cdot|_{L^p}$ and $(\cdot,\cdot) = (\cdot,\cdot)_{L^2}$.

The spherical gradient, Laplacian, curl and covariant derivative are given by

$$\begin{aligned}
\nabla u &= \partial_\theta u\,\boldsymbol{e}_\theta + \frac{1}{\sin\theta}\,\partial_\phi u\,\boldsymbol{e}_\phi,\\
\Delta u &= \frac{1}{\sin^2\theta}\,\partial^2_{\phi\phi} u + \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta u),\\
(\nabla\times\boldsymbol{u})\cdot\boldsymbol{e}_r &= (\partial_\theta(\sin\theta\,u_\phi) - \partial_\phi u_\theta)/\sin\theta)\,\boldsymbol{e}_r,\\
\nabla_{\boldsymbol{u}}\boldsymbol{v} &= (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\\
&= (u_\theta\,\partial_\theta v_\theta + u_\phi\partial_\phi v_\theta/\sin\theta)\,\boldsymbol{e}_\theta + (u_\theta\,\partial_\theta v_\phi + u_\phi\,\partial_\phi v_\phi/\sin\theta)\,\boldsymbol{e}_\phi,
\end{aligned}$$

where $\boldsymbol{e}_\theta = \cos\theta\cos\phi\,\boldsymbol{e}_x + \cos\theta\sin\phi\,\boldsymbol{e}_y - \sin\theta\,\boldsymbol{e}_z$, $\boldsymbol{e}_\phi = -\sin\phi\,\boldsymbol{e}_x + \cos\phi\,\boldsymbol{e}_y$ and $\boldsymbol{e}_r = \sin\theta\cos\phi\,\boldsymbol{e}_x + \sin\theta\sin\phi\,\boldsymbol{e}_y + \cos\theta\,\boldsymbol{e}_z$ are the unit vectors in the corresponding directions. Thus integration by parts leads to the following identity:

$$\begin{aligned}
(u,-\Delta u)_{L^2} &= -\int_0^\pi\int_0^{2\pi} \frac{1}{\sin\theta}\,u\,\overline{\partial^2_{\phi\phi}u}\,\mathrm{d}\phi\,\mathrm{d}\theta - \int_0^\pi\int_0^{2\pi} u\,\overline{\partial_\theta(\sin\theta\,\partial_\theta u)}\,\mathrm{d}\phi\,\mathrm{d}\theta\\
&= \int_0^\pi\int_0^{2\pi}\frac{1}{\sin\theta}|\partial_\phi u|^2\,\mathrm{d}\phi\,\mathrm{d}\theta + \int_0^\pi\int_0^{2\pi}\sin\theta|\partial_\theta u|^2\,\mathrm{d}\phi\,\mathrm{d}\theta\\
&= |\nabla u|^2_{L^2} = |(-\Delta)^{1/2}u|^2_{L^2}.
\end{aligned}\tag{4.1.1}$$

The Jacobian is given by

$$\partial(\psi,\omega) = \frac{1}{\sin\theta}(\partial_\theta\psi\,\partial_\phi\omega - \partial_\phi\psi\,\partial_\theta\omega),\tag{4.1.2}$$

which, as in the planar case, satisfies the following properties:

$$\begin{aligned}
(\partial(a,b),c) &= \int_0^\pi\int_0^{2\pi}(-\partial_\phi(a\,\partial_\theta b) + \partial_\theta(a\,\partial_\phi b))\,c\,\mathrm{d}\phi\,\mathrm{d}\theta\\
&= \int_0^\pi\int_0^{2\pi}(a\,\partial_\theta b\,\partial_\phi c - a\,\partial_\phi b\,\partial_\theta c)\,\mathrm{d}\phi\,\mathrm{d}\theta \qquad \text{by integration by parts}\\
&= (\partial(b,c),a) = (\partial(c,a),b) \qquad \text{by symmetry,}
\end{aligned}\tag{4.1.3}$$

and

$$
\begin{aligned}
(\partial(a,b),b) &= (\partial(b,b),a) && \text{by (4.1.3)} \\
&= 0 && \text{by (4.1.2)},
\end{aligned} \tag{4.1.4}
$$

for all real a, b and c such that their integrals over S^2 vanish and the expressions above are defined. We also note that for a, b such that $\partial_\phi a = \partial_\phi b = 0$, the Jacobian simplifies to

$$
\partial(a,b) = \frac{1}{\sin\theta}(\partial_\theta a\, \partial_\phi b - \partial_\phi a\, \partial_\theta b) = 0. \tag{4.1.5}
$$

Analogous to Fourier expansion in the planar case, we expand u using the spherical harmonics Y_{lm}, following the conventions given in [21]:

$$
u(\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} u_{lm}(t) Y_{lm}(\theta,\phi), \tag{4.1.6}
$$

where Y_{lm} is defined by

$$
Y_{lm}(\theta,\phi) = \left(\frac{(l-m)!(2l+1)}{4\pi(l+m)!}\right)^{1/2} \mathrm{e}^{im\phi} P_l^m(\cos\theta) \tag{4.1.7}
$$

and the associated Legendre polynomials P_l^m are solutions to

$$
(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_l^m(x) - 2x\frac{\mathrm{d}}{\mathrm{d}x}P_l^m(x) + \left(l(l+1) - \frac{m^2}{1-x^2}\right)P_l^m(x) = 0.
$$

The coefficients u_{lm} of (4.1.6) are given by

$$
u_{lm}(t) := \int_{S^2} u(\theta,\phi,t)\overline{Y_{lm}}(\theta,\phi)\,\mathrm{d}A,
$$

where $(\bar{\ })$ denotes the complex conjugate. The operator $-\Delta$ has Y_{lm} as its eigenfunctions with corresponding eigenvalues $l(l+1)$:

$$
-\Delta Y_{lm} = l(l+1)Y_{lm}, \tag{4.1.8}
$$

which implies that Y_{lm} form an orthonormal basis of $L^2(S^2)$ (see [22]). Hence this justifies the expansion in (4.1.6), because of the completeness and orthonormality of

Y_{lm}:

$$\int_0^{\pi}\int_0^{2\pi} Y_{l_1m_1}(\theta,\phi)\overline{Y_{l_2m_2}}(\theta,\phi)\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta = \delta_{l_1,l_2}\delta_{m_1,m_2}, \tag{4.1.9}$$

where δ is the Kronecker delta.

By definition, one immediately sees that

$$\partial_\phi\omega = \partial_\phi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\omega_{lm}Y_{lm} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\omega_{lm}\partial_\phi Y_{lm} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\mathrm{i}m\,\omega_{lm}Y_{lm}.$$

Using Y_{lm} and (4.1.9), the inner product becomes

$$\begin{aligned}(u,v) &= \int_0^{\pi}\int_0^{2\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}u_{lm}Y_{lm}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\overline{v_{lm}}\overline{Y_{lm}}\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}u_{lm}\overline{v_{lm}}\end{aligned} \tag{4.1.10}$$

for $u,\ v \in L^2(S^2)$, analogous to the Cartesian case. By (4.1.1) and (4.1.8), this implies that

$$|\nabla u|^2 = (u,-\Delta u) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}l(l+1)|u_{lm}|^2. \tag{4.1.11}$$

Using spherical expansion, we obtain

$$\begin{aligned}|\partial^2_{\theta\phi}v|^2 &\le |(-\Delta)^{1/2}\partial_\phi v|^2\\ &= -\int_{S^2}\partial_\phi v\cdot\overline{\Delta\partial_\phi v}\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta && \text{by integration by parts}\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}m^2|v_{lm}|^2\,l(l+1) && \text{by the orthogonality of } Y_{lm}\\ &\le \sum_{l=0}^{\infty}\sum_{m=-l}^{l}l^2(l+1)^2|v_{lm}|^2 = |\Delta v|^2.\end{aligned} \tag{4.1.12}$$

Finally, to conclude our collection of basic spherical properties, the Poincaré constant κ_0 is exactly $\sqrt{2}$ by definition:

$$\begin{aligned}\kappa_0 &:= \inf_u\frac{|\nabla u|}{|u|} = \inf_u\Big(\Big(\sum_{l=0}^{\infty}\sum_{m=-l}^{l}l(l+1)|u_{lm}|^2\Big)/\Big(\sum_{l=0}^{\infty}\sum_{m=-l}^{l}|u_{lm}|^2\Big)\Big)^{1/2}\\ &= \sqrt{2}.\end{aligned}$$

4.2 Statement of the problem

We recall the Navier–Stokes equations on the rotating sphere:

$$\partial_t \boldsymbol{v} + \nabla_{\boldsymbol{v}} \boldsymbol{v} + \frac{2}{\varepsilon} \cos\theta\, \boldsymbol{v}^\perp + \nabla p = \mu \Delta \boldsymbol{v} + f_{\boldsymbol{v}}, \tag{4.2.1}$$

$$\nabla \cdot \boldsymbol{v} := \frac{1}{\sin\theta}(\partial_\theta(\sin\theta\, v_\theta) + \partial_\phi v_\phi) = 0,$$

where $1/\varepsilon$ is the angular velocity at which the sphere rotates about a fixed axis and $\boldsymbol{v}^\perp$ is $\boldsymbol{v}$ rotated by $\pi/2$.

By Hodge's decomposition theorem (see [19]), $\boldsymbol{v}$ can be written as

$$\boldsymbol{v} = \nabla\varphi + \nabla^\perp\psi, \tag{4.2.2}$$

where φ and ψ are scalars. Taking the divergence of (4.2.2) gives

$$\begin{aligned} 0 = \nabla \cdot \boldsymbol{v} &= \nabla \cdot \nabla\varphi + \nabla \cdot \nabla^\perp \psi \\ &= \Delta\varphi, \end{aligned} \tag{4.2.3}$$

and by expansion in spherical harmonics,

$$\Delta\varphi(\theta, \phi, t) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} l(l+1)\varphi_{lm}(t) Y_{lm}(\theta, \phi) = 0, \tag{4.2.4}$$

which implies that $\varphi_{lm}(t) = 0$ for $l \neq 0$. Thus $\varphi(\theta, \phi, t) = \varphi_{00}(t)Y_{00}$, implying that $\nabla\varphi = 0$.

Now, taking the curl (i.e. $\nabla^\perp \cdot$) of (4.2.2) gives

$$\omega := \nabla^\perp \cdot \boldsymbol{v} = \nabla^\perp \cdot \nabla^\perp \psi = \Delta\psi \tag{4.2.5}$$

(where we fix ψ uniquely by requiring that $\int_{S^2} \psi = 0$). Therefore, any sufficiently smooth and divergence-free $\boldsymbol{v}$ can be written as $\boldsymbol{v} = \nabla^\perp\psi = \nabla^\perp\Delta^{-1}\omega$.

Turning to (4.2.1), we take its curl to obtain the vorticity form:

$$\partial_t \omega + \partial(\psi, \omega) + \frac{2}{\varepsilon}\, \partial_\phi \psi = \mu\Delta\omega + f, \tag{4.2.6}$$

where $f := \nabla^\perp \cdot f_{\boldsymbol{v}}$ necessarily has 0 integral over S^2.

We define the determining modes on the sphere analogously to the planar case. As in (3.1.28), we consider two solutions ω, $\omega^\sharp$ (with corresponding streamfunctions ψ, $\psi^\sharp$) of (4.2.1), with the same f but different initial conditions:

$$\partial_t\omega + \partial(\psi,\omega) + \frac{2}{\varepsilon}\,\partial_\phi\psi = \mu\Delta\omega + f, \tag{4.2.7}$$
$$\partial_t\omega^\sharp + \partial(\psi^\sharp,\omega^\sharp) + \frac{2}{\varepsilon}\,\partial_\phi\psi^\sharp = \mu\Delta\omega^\sharp + f. \tag{4.2.8}$$

By defining $\delta\omega := \omega - \omega^\sharp$ and $\delta\psi := \psi - \psi^\sharp$, we have

$$\begin{aligned}\partial(\psi,\omega) - \partial(\psi^\sharp,\omega^\sharp) &= \frac{1}{\sin\theta}\left(\partial_\theta\psi^\sharp\,\partial_\phi\delta\omega - \partial_\phi\psi^\sharp\,\partial_\theta\delta\omega + \partial_\theta\delta\psi\,\partial_\phi\omega - \partial_\phi\delta\psi\,\partial_\theta\omega\right)\\ &= \partial(\psi^\sharp,\delta\omega) + \partial(\delta\psi,\omega).\end{aligned} \tag{4.2.9}$$

Subtracting (4.2.8) from (4.2.7) thus gives

$$\partial_t\delta\omega + \partial(\psi^\sharp,\delta\omega) + \partial(\delta\psi,\omega) + \frac{2}{\varepsilon}\,\partial_\phi\delta\psi = \mu\Delta\delta\omega. \tag{4.2.10}$$

Then, by fixing a threshold wavenumber $\kappa \geq \kappa_0$, we define P_κ as the L^2 projection to lower modes:

$$\delta\omega^<(\theta,\phi,t) := \mathsf{P}_\kappa\delta\omega(\theta,\phi,t) := \sum_{l\leq\kappa}\sum_{m=-l}^{l}\delta\omega_{lm}(t)Y_{lm}(\theta,\phi), \tag{4.2.11}$$

and the projection to higher modes by

$$\delta\omega^>(\theta,\phi,t) := \delta\omega(\theta,\phi,t) - \delta\omega^<(\theta,\phi,t) = \sum_{l>\kappa}\sum_{m=-l}^{l}\delta\omega_{lm}(t)Y_{lm}(\theta,\phi). \tag{4.2.12}$$

Using the definition of P_κ, we obtain the following Poincaré-type inequalities:

$$|\delta\omega^>|^2 = \sum_{l>\kappa}\sum_{m=-l}^{l}|\delta\omega_{lm}|^2 \leq \sum_{l>\kappa}\sum_{m=-l}^{l}\frac{l(l+1)}{\kappa^2}\,|\delta\omega_{lm}|^2 = \frac{1}{\kappa^2}|\nabla\delta\omega^>|^2,$$

which gives

$$\kappa|\delta\omega^>| \leq |\nabla\delta\omega^>|, \tag{4.2.13}$$

and

$$|\nabla\delta\omega^<|^2 = \sum_{l\leq\kappa}\sum_{m=-l}^{l}l(l+1)\,|\delta\omega_{lm}|^2 \leq \kappa(\kappa+1)\sum_{l\leq\kappa}\sum_{m=-l}^{l}|\delta\omega_{lm}|^2 = \kappa(\kappa+1)|\delta\omega^<|^2,$$

leading to

$$|\nabla\delta\omega^{<}|^2 \leq c_? \, \kappa^2 |\delta\omega^{<}|^2, \tag{4.2.14}$$

where $c_? = 1 + \kappa_0^{-1} = 1 + 1/\sqrt{2}$.

Our aim for this chapter is to obtain an improved bound on the existing general case result (3.1.34) on the rotating sphere, i.e. to find a tighter bound on the threshold wavenumber κ such that $|\delta\omega^{<}(t)| \to 0$ implies $|\delta\omega(t)| \to 0$. Analogously to Chapter 3, we separate the vorticity into its zonal (zero frequency) and non-zonal components, which we define by

$$\bar{\omega}(\theta, t) := \frac{1}{2\pi}\int_0^{2\pi} \omega(\theta, \phi, t)\,\mathrm{d}\phi, \qquad \text{and} \tag{4.2.15}$$

$$\tilde{\omega}(\theta, \phi, t) := \omega(\theta, \phi, t) - \bar{\omega}(\theta, t). \tag{4.2.16}$$

Using spherical harmonics, these are expressed as

$$\bar{\omega}(\theta, t) = \sum_{l=0}^{\infty} \omega_{l0}(\theta, t) Y_{l0}(\theta, \phi), \qquad \text{and} \tag{4.2.17}$$

$$\tilde{\omega}(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq 0}}^{l} \omega_{lm}(\theta, \phi, t) Y_{lm}(\theta, \phi), \tag{4.2.18}$$

since $\bar{\omega}$ being independent of ϕ implies that all $m \neq 0$ terms in the expansion must be 0. For convenience and consistency, we write

$$\bar{\omega}_{lm} = \begin{cases} \omega_{lm} & m = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.19}$$

and

$$\tilde{\omega}_{lm} = \begin{cases} 0 & m = 0 \\ \omega_{lm} & \text{otherwise.} \end{cases} \tag{4.2.20}$$

Thus $\bar{\omega}$ and $\tilde{\omega}$ are orthogonal in H^s for $s = 1, 2, \cdots$:

$$(\bar{\omega}, \tilde{\omega})_{H^s} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l(l+1))^s \, \bar{\omega}_{lm} \, \overline{\tilde{\omega}_{lm}}$$

$$= \sum_{l=0}^{\infty} (l(l+1))^s \bar{\omega}_{l0}\,\overline{\tilde{\omega}_{l0}} + \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq 0}}^{l} (l(l+1))^s\,\bar{\omega}_{lm}\,\overline{\tilde{\omega}_{lm}}$$

$$= 0. \tag{4.2.21}$$

With $\bar{\omega}$ and $\tilde{\omega}$ thus defined, we state the main result by Wirosoetisno [8], using our definition of $\mathcal{G}_m$. Recall that $\nu_0 = \mu\kappa_0^2$.

Theorem 14. *Assume that the initial data $\boldsymbol{v}(0) \in L^2(S^2)$ and that $|\Delta f|_{L^2(S^2)} < \infty$. Then there exists a time $\mathcal{T}_0(|\boldsymbol{v}(0)|_{L^2(S^2)})$ and a constant $c_9(\nu_0)$ such that*

$$|\tilde{\omega}(t)|^2_{L^2(S^2)} + \mu \int_t^{t+1} |\nabla\tilde{\omega}(\tau)|^2_{L^2(S^2)}\,\mathrm{d}\tau \le \varepsilon M_0/\kappa_0^2, \tag{4.2.22}$$

$$|\tilde{\omega}(t)|^2_{L^2(S^2)} + \mu \int_0^{t} |\nabla\tilde{\omega}(\tau)|^2_{L^2(S^2)} \mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau \le \varepsilon M_0/\kappa_0^2 \tag{4.2.23}$$

for all $t \ge \mathcal{T}_0$, where

$$M_0 = c_9\,\mathcal{G}_2\mathcal{G}_3(1+\mathcal{G}_0^2). \tag{4.2.24}$$

Again, the constants in [8] may include lengths, whereas ours are dimensionless, which accounts for the extra factor of κ_0^{-2}.

4.3 Consequences of different forms of forcing

For the purposes of this chapter, we consider the below forms of zonal forcing, modified from those mentioned in Chapter 3 for the sphere.

$$\text{Bandwidth-limited:} \quad \bar{f} = \mathsf{P}_{\kappa_f}\bar{f} \qquad (\kappa_f \ge \kappa_0), \tag{4.3.1}$$

$$\text{Algebraic decay:} \quad |\bar{f}_{l0}| \le \frac{\nu_0^2\,\kappa_0^{s-1}\,(l(l+1))^{-s/2}}{\sqrt{2}\zeta(2+2s)^{1/2}}\,\mathcal{G}_0 \qquad (s > 5/2), \tag{4.3.2}$$

$$\text{Exponential decay:} \quad |\bar{f}_{l0}| \le \frac{\nu_0^2}{\sqrt{2}\,\kappa_0}\Big(\frac{2\gamma}{1+2\gamma}\Big)^{1/2} \mathrm{e}^{\gamma(1-l/\kappa_0)}\,\mathcal{G}_0 \quad (\gamma > 0), \tag{4.3.3}$$

where $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. Again, the requirement for (4.3.2) that $s > 5/2$ is purely to ensure that $\bar{f} \in H^2(S^2)$, so that we can apply Theorem 14. The precise expressions for (4.3.2) and (4.3.3) have been chosen to ensure that $|\nabla^{-1}\bar{f}|/(\mu\kappa_0)^2 \le \mathcal{G}_0$, in order to be consistent with the definition of the Grashof number given in (2.1.5):

$$|\bar{f}_{l0}| \le \frac{\nu_0^2\,\kappa_0^{s-1}\,(l(l+1))^{-s/2}}{\sqrt{2}\zeta(2+2s)^{1/2}}\,\mathcal{G}_0 \qquad \text{implies}$$

$$\begin{aligned}
|\nabla^{-1}\bar{f}|_2^2 &= \sum_{l=1}^{\infty}\frac{|f_{l0}|^2}{l(l+1)} \le \sum_{l=1}^{\infty}\frac{\nu_0^4\,\kappa_0^{2s-2}\,\mathcal{G}_0^2}{2\zeta(2+2s)(l(l+1))^{s+1}}\\
&= \frac{\nu_0^4\,\kappa_0^{2s-2}\,\mathcal{G}_0^2}{2\zeta(2+2s)}\kappa_0^{-2s-2} + \frac{\nu_0^4\,\kappa_0^{2s-2}\,\mathcal{G}_0^2}{2\zeta(2+2s)}\sum_{l=2}^{\infty}(l(l+1))^{-s-1}\\
&\le \frac{\mu^4\kappa_0^4\,\mathcal{G}_0^2}{2\zeta(2+2s)} + \frac{\nu_0^4\,\kappa_0^{2s-3}\,\mathcal{G}_0^2}{2\zeta(2+2s)}\int_{\kappa_0}^{\infty}(k(k+1))^{-s-1}\,\mathrm{d}k\\
&\le \frac{\mu^4\kappa_0^4\,\mathcal{G}_0^2}{2\zeta(2+2s)} + \frac{\nu_0^4\,\kappa_0^{2s-3}\,\mathcal{G}_0^2}{2\zeta(2+2s)}\int_{\kappa_0}^{\infty}k^{-2s-2}\,\mathrm{d}k\\
&= \frac{\mu^4\,\kappa_0^4\,\mathcal{G}_0^2}{2\zeta(2+2s)} + \frac{\mu^4\kappa_0^4\,\mathcal{G}_0^2}{2(2s+1)\zeta(2+2s)}\\
&\le \frac{\mu^4\kappa_0^4\,\mathcal{G}_0^2}{\zeta(2+2s)} \qquad\qquad \text{since } s > 5/2,
\end{aligned} \tag{4.3.4}$$

where the sum in the first line has been taken from $l = 1$ because our assumption that $\int_{S^2} f = 0$ implies that $f_{00} = 0$, by definition. Rearranging (4.3.4) thus gives

$$|\nabla^{-1}\bar{f}|/(\mu\kappa_0)^2 \le \Big(\zeta(2+2s)\Big)^{-1/2}\mathcal{G}_0 \le \mathcal{G}_0,$$

since ζ is decreasing in s and $\lim_{s\to\infty}\zeta(s) = 1$. We also check that

$$|\bar{f}_{l0}| \le \frac{\nu_0^2}{\sqrt{2}\,\kappa_0}\left(\frac{2\gamma}{1+2\gamma}\right)^{1/2}\mathrm{e}^{\gamma(1-l/\kappa_0)}\,\mathcal{G}_0 \qquad \text{implies}$$

$$\begin{aligned}
|\nabla^{-1}\bar{f}|^2 &= \sum_{l=1}^{\infty}\frac{|f_{l0}|^2}{l(l+1)}\\
&\le \frac{\nu_0^4\,\mathcal{G}_0^2}{\kappa_0^2}\left(\frac{\gamma}{1+2\gamma}\right)\mathrm{e}^{2\gamma}\sum_{l=1}^{\infty}\frac{\mathrm{e}^{-2\gamma l/\kappa_0}}{l(l+1)}\\
&\le \mu^4\kappa_0^4\,\mathcal{G}_0^2\left(\frac{\gamma}{1+2\gamma}\right)\mathrm{e}^{2\gamma}\sum_{l=1}^{\infty}\mathrm{e}^{-2\gamma l/\kappa_0} \qquad \text{since } (l(l+1))^{-1} \le \frac{1}{2} = \kappa_0^{-2}\\
&\le \mu^4\kappa_0^4\,\mathcal{G}_0^2\left(\frac{\gamma}{1+2\gamma}\right)(1+\mathrm{e}^{-2\gamma}+\mathrm{e}^{-4\gamma}+\cdots) \quad \text{since } \kappa_0^{-1} < 1\\
&= \mu^4\kappa_0^4\,\mathcal{G}_0^2\left(\frac{\gamma}{1+2\gamma}\right)/(1-\mathrm{e}^{-2\gamma}) \le \mu^4\kappa_0^4\,\mathcal{G}_0^2 \qquad \text{for all } \gamma > 0,
\end{aligned}$$

which implies that

$$|\nabla^{-1}\bar{f}|/(\mu\kappa_0)^2 \le \mathcal{G}_0.$$

With these forms of $\bar{f}$ in mind, we state and prove the following intermediate results.

Lemma 15. *Suppose ω satisfies (4.2.6) and define $\bar{\omega}^{<f} := \mathsf{P}_{\kappa_f}\bar{\omega}$, $\bar{\omega}^{>f} := \bar{\omega} - \bar{\omega}^{<f}$ for some $\kappa_f \ge \kappa_0$. Assume $\nu_0 = \mu\kappa_0^2 < 1$. Then there exists an absolute constant c_{**} such that*

(a) if $\bar{f}$ satisfies (4.3.1), then

$$\int_0^t |\nabla\bar{\omega}^{>f}|^2_{L^2(S^2)} \mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau \le 3\,c_{**}(\varepsilon M_0)^2/\nu_0^3\,; \tag{4.3.5}$$

(b) if $\bar{f}$ satisfies (4.3.2), then

$$\int_0^t |\nabla\bar{\omega}^{>f}|^2_{L^2(S^2)} \mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau \le \quad c_{**}(\varepsilon M_0)^2/\nu_0^3 + \frac{4\,\nu_0}{(2s+1)\zeta(2s+2)}\left(\frac{\kappa_0}{\kappa_f}\right)^{2s+1}\mathcal{G}_0^2, \quad or \tag{4.3.6}$$

(c) if $\bar{f}$ satisfies (4.3.3),

$$\int_0^t |\nabla\bar{\omega}^{>f}|^2_{L^2(S^2)} \mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau \le \quad c_{**}(\varepsilon M_0)^2/\nu_0^3 + 8\,\nu_0\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\,\mathcal{G}_0^2. \tag{4.3.7}$$

Proof. We first remark that conceptually, this proof is analogous to that of Lemma 12. The differences are purely down to the individual results and inequalities used having planar and spherical versions (for example, Lemma 14 being the spherical analogue of Lemma 11), and the types of forcing being defined differently. We therefore omit individual technical details to avoid complete repetition.

We begin by multiplying (4.2.7) by $\bar{\omega}^{>f}$ in L^2:

$$(\partial_t\omega, \bar{\omega}^{>f}) + (\partial(\psi,\omega), \bar{\omega}^{>f}) + \frac{2}{\varepsilon}(\partial_\phi\psi, \bar{\omega}^{>f}) = \mu(\Delta\omega, \bar{\omega}^{>f}) + (f, \bar{\omega}^{>f}). \tag{4.3.8}$$

The first term becomes

$$(\partial_t\omega, \bar{\omega}^{>f}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\omega}^{>f}|^2 \tag{4.3.9}$$

by the orthogonality of $\bar{\omega}^{<f}$ and $\bar{\omega}^{>f}$ and (4.2.21). By splitting $\omega = \bar{\omega} + \tilde{\omega}$ and

$\psi = \bar{\psi} + \tilde{\psi}$, the second term of (4.3.8) becomes

$$\begin{aligned}(\partial(\psi,\omega),\bar{\omega}^{>f}) &= (\partial(\psi,\tilde{\omega}),\bar{\omega}^{>f}) + (\partial(\psi,\bar{\omega}),\bar{\omega}^{>f})\\ &= (\partial(\tilde{\psi},\tilde{\omega}),\bar{\omega}^{>f}).\end{aligned} \tag{4.3.10}$$

Integration by parts shows that the third term of (4.3.8) is 0:

$$\begin{aligned}\frac{2}{\varepsilon}(\partial_\phi\psi,\bar{\omega}^{>f}) &= \frac{2}{\varepsilon}(\partial_\phi\tilde{\psi},\bar{\omega}^{>f}) + \frac{2}{\varepsilon}(\partial_\phi\bar{\psi},\bar{\omega}^{>f}) = \frac{2}{\varepsilon}(\partial_\phi\tilde{\psi},\bar{\omega}^{>f}) \quad \text{since } \partial_\phi\bar{\psi} = 0\\ &= \frac{2}{\varepsilon}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathrm{i}m\,\tilde{\psi}_{lm}Y_{lm}\overline{\bar{\omega}^{>f}_{lm}}\overline{Y_{lm}}\\ &= 0 \qquad\qquad \text{by (4.2.19), (4.2.20).}\end{aligned} \tag{4.3.11}$$

The first term on the right hand side of (4.3.8) becomes

$$\begin{aligned}\mu(\Delta\omega,\bar{\omega}^{>f}) &= \mu(\Delta\bar{\omega},\bar{\omega}^{>f}) + \mu(\Delta\tilde{\omega},\bar{\omega}^{>f})\\ &= \mu(\Delta\bar{\omega},\bar{\omega}^{>f}) && \text{by (4.2.21)}\\ &= \mu(\Delta\bar{\omega}^{<f},\bar{\omega}^{>f}) + \mu(\Delta\bar{\omega}^{>f},\bar{\omega}^{>f})\\ &= \mu(\Delta\bar{\omega}^{>f},\bar{\omega}^{>f}) && \text{by the orthogonality of } \bar{\omega}^{<f} \text{ and } \bar{\omega}^{>f}\\ &= -\mu|\nabla\bar{\omega}^{>f}|^2 && \text{by (4.1.1).}\end{aligned} \tag{4.3.12}$$

Collecting (4.3.9) to (4.3.12) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{\omega}^{>f}|^2 + \mu|\nabla\bar{\omega}^{>f}|^2 = -(\partial(\tilde{\psi},\tilde{\omega}),\bar{\omega}^{>f}) + (f,\bar{\omega}^{>f}). \tag{4.3.13}$$

Assuming t is large enough, we apply Theorem 14 so that the first term on the right hand side of this can be bounded as

$$\begin{aligned}|(\partial(\tilde{\psi},\tilde{\omega}),\bar{\omega}^{>f})| &= |(\partial(\bar{\omega}^{>f},\tilde{\psi}),\tilde{\omega})| && \text{by (4.1.3)}\\ &\le |\nabla\tilde{\psi}|_4|\tilde{\omega}|_4|\nabla\bar{\omega}^{>f}|_2 && \text{by Hölder}\\ &\le \frac{2}{\mu}|\nabla\tilde{\psi}|_4^2|\tilde{\omega}|_4^2 + \frac{\mu}{8}|\nabla\bar{\omega}^{>f}|^2 && \text{by Young}\\ &\le \frac{c}{\mu}|\nabla\tilde{\psi}||\tilde{\omega}|^2|\nabla\tilde{\omega}| + \frac{\mu}{8}|\nabla\bar{\omega}^{>f}|^2 && \text{by Ladyzhenskaya}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{c}{\nu_0}\,\varepsilon M_0 |\nabla\tilde\psi||\nabla\tilde\omega| + \frac{\mu}{8}|\nabla\bar\omega^{>f}|^2 && \text{by (4.2.22)}\\
&\le c\,\frac{\varepsilon M_0}{\nu_0\kappa_0^2}|\nabla\tilde\omega|^2 + \frac{\mu}{8}|\nabla\bar\omega^{>f}|^2 && \text{by Poincaré.} \qquad (4.3.14)
\end{aligned}
$$

Similarly, we bound the forcing term by

$$
(f,\bar\omega^{>f}) \le \frac{2}{\mu}|\nabla^{-1}\bar f^{>f}|^2 + \frac{\mu}{8}|\nabla\bar\omega^{>f}|^2. \tag{4.3.15}
$$

Thus (4.3.13) becomes

$$
\frac{\mathrm{d}}{\mathrm{d}t}|\bar\omega^{>f}|^2 + \frac{3}{2}\mu|\nabla\bar\omega^{>f}|^2 \le c\,\frac{\varepsilon M_0}{\nu_0\kappa_0^2}|\nabla\tilde\omega|^2 + \frac{4}{\mu}|\nabla^{-1}\bar f^{>f}|^2. \tag{4.3.16}
$$

We use the Poincaré-type inequality (4.2.13) on the $|\nabla\bar\omega^{>f}|$ term on the left hand side:

$$
\frac{\mathrm{d}}{\mathrm{d}t}|\bar\omega^{>f}|^2 + \nu_0\,|\bar\omega^{>f}|^2 + \frac{\mu}{2}|\nabla\bar\omega^{>f}|^2 \le c\,\frac{\varepsilon M_0}{\nu_0\kappa_0^2}|\nabla\tilde\omega|^2 + \frac{4}{\mu}|\nabla^{-1}\bar f^{>f}|^2 \tag{4.3.17}
$$

and multiply by $\mathrm{e}^{\nu_0 t}$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\nu_0 t}|\bar\omega^{>f}|^2) + \frac{\mu}{2}\,\mathrm{e}^{\nu_0 t}|\nabla\bar\omega^{>f}|^2 \le c\,\frac{\varepsilon M_0}{\nu_0\kappa_0^2}|\nabla\tilde\omega|^2\mathrm{e}^{\nu_0 t} + \frac{4\,\mathrm{e}^{\nu_0 t}}{\mu}|\nabla^{-1}\bar f^{>f}|^2. \tag{4.3.18}
$$

We then integrate in time and multiply by $\mathrm{e}^{-\nu_0 t}$:

$$
\begin{aligned}
&|\bar\omega^{>f}(t)|^2 + \frac{\mu}{2}\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar\omega^{>f}|^2\,\mathrm{d}\tau\\
&\qquad\le \mathrm{e}^{-\nu_0 t}|\bar\omega^{>f}(0)|^2 + c\,\frac{\varepsilon M_0}{\nu_0\kappa_0^2}\int_0^t |\nabla\tilde\omega|^2\mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau + \frac{4}{\mu\nu_0}|\nabla^{-1}\bar f^{>f}|^2\\
&\qquad\le \frac{c_*(\varepsilon M_0)^2}{2\,\nu_0^2\kappa_0^2} + \frac{4}{\mu\nu_0}|\nabla^{-1}\bar f^{>f}|^2,
\end{aligned} \tag{4.3.19}
$$

where we have used (4.2.23) and assumed t is large enough for the adjusted constant c_* to absorb the $|\bar\omega^{>f}(0)|^2$ term into the $(\varepsilon M_0)^2$ term.

We now consider the consequences of the hypotheses (4.3.1) to (4.3.3). When $\bar f$ satisfies (4.3.1), we have $\bar f^{>f} = 0$ by definition, so that (4.3.19) becomes

$$
\frac{\mu}{2}\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar\omega^{>f}|^2\,\mathrm{d}\tau \le \frac{c_*(\varepsilon M_0)^2}{2\,\nu_0^2\kappa_0^2}, \tag{4.3.20}
$$

where we have used Lemma 9 and dropped the first term on the left hand side. With our assumption that $\nu_0 < 1$ and hence $\mathrm{e}^{\nu_0} < 3$, we arrive at

$$\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le 3\,c_*(\varepsilon M_0)^2/\nu_0^3, \tag{4.3.21}$$

which is (4.3.5).

When $\bar{f}$ satisfies (4.3.2) instead, we have

$$\begin{aligned}
|\nabla^{-1}\bar{f}^{>f}|^2 &= \sum_{l>\kappa_f}^{\infty}\frac{|f_{l0}|^2}{l(l+1)} \le \sum_{l>\kappa_f}^{\infty}\frac{\nu_0^4\kappa_0^{2s-2}(l(l+1))^{-(s+1)}}{2\zeta(2+2s)}\,\mathcal{G}_0^2\\
&= \frac{\nu_0^4\kappa_0^{2s-2}}{2\zeta(2+2s)}\,\mathcal{G}_0^2\sum_{l>\kappa_f}^{\infty}(l(l+1))^{-(s+1)}\\
&\le \frac{\nu_0^4\kappa_0^{2s-3}}{2\zeta(2+2s)}\,\mathcal{G}_0^2\int_{\kappa_f}^{\infty}\frac{\mathrm{d}l}{(l(l+1))^{s+1}}\\
&\qquad\qquad\text{(since terms in the sum are non-negative and decreasing)}\\
&\le \frac{\nu_0^4\kappa_0^{2s-3}}{2\zeta(2+2s)}\,\mathcal{G}_0^2\int_{\kappa_f}^{\infty}\frac{\mathrm{d}l}{l^{2s+2}} = \frac{\mu^4\kappa_0^4\,(\kappa_0/\kappa_f)^{2s+1}}{2(2s+1)\zeta(2+2s)}\,\mathcal{G}_0^2,
\end{aligned} \tag{4.3.22}$$

so after ignoring the first term on the left hand side, (4.3.19) becomes

$$\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le c_*(\varepsilon M_0)^2/\nu_0^3 + \frac{4\,\nu_0(\kappa_0/\kappa_f)^{2s+1}}{(2s+1)\zeta(2s+2)}\mathcal{G}_0^2, \tag{4.3.23}$$

which is (4.3.6).

Finally, when $\bar{f}$ satisfies (4.3.3),

$$\begin{aligned}
|\nabla^{-1}\bar{f}^{>f}|^2 &= \sum_{l>\kappa_f}^{\infty}\frac{|f_{l0}|^2}{l(l+1)} \le \sum_{l>\kappa_f}^{\infty}\frac{\nu_0^4}{2\kappa_0^2}\Big(\frac{2\gamma}{1+2\gamma}\Big)\frac{\mathrm{e}^{2\gamma(1-l/\kappa_0)}}{l(l+1)}\,\mathcal{G}_0^2\\
&\le \sum_{l>\kappa_f}^{\infty}\frac{\nu_0^4}{2\kappa_0^4}\Big(\frac{2\gamma}{1+2\gamma}\Big)\mathrm{e}^{2\gamma(1-l/\kappa_0)}\,\mathcal{G}_0^2 \qquad \text{since } l(l+1)>\kappa_0^2\\
&= \frac{\nu_0^4}{\kappa_0^4}\Big(\frac{\gamma}{1+2\gamma}\Big)\,\mathcal{G}_0^2\sum_{l>\kappa_f}^{\infty}\mathrm{e}^{2\gamma(1-l/\kappa_0)}\\
&\le \frac{\nu_0^4}{\kappa_0^4}\Big(\frac{\gamma}{1+2\gamma}\Big)\,\mathcal{G}_0^2\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}(1+\mathrm{e}^{-2\gamma}+\mathrm{e}^{-4\gamma}+\cdots)\\
&= \frac{\nu_0^4}{\kappa_0^4}\Big(\frac{\gamma}{1+2\gamma}\Big)\,\mathcal{G}_0^2\,\frac{\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}}{1-\mathrm{e}^{-2\gamma}} \le \frac{\nu_0^4}{\kappa_0^4}\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\,\mathcal{G}_0^2,
\end{aligned} \tag{4.3.24}$$

so (4.3.19) becomes

$$\int_0^t \mathrm{e}^{\nu_0(\tau-t)}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le c_*(\varepsilon M_0)^2/\nu_0^3 + 8\,\nu_0\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\mathcal{G}_0^2, \tag{4.3.25}$$

giving (4.3.7). □

4.4 Bounds on the number of determining modes

We now state the main theorem for this chapter. We assume, as with Lemma 15, that $\nu_0 < 1$.

Theorem 16 (Determining modes on the sphere)**.** *Let $\delta\omega$ be the solution of* (4.2.10) *with $f \in H^2(S^2)$. Then the low modes are determining, i.e.* $\lim_{t\to\infty} |\mathsf{P}_\kappa \delta\omega(t)|_{L^2(S^2)} = 0$ *implies that* $\lim_{t\to\infty} |\delta\omega(t)|_{L^2(S^2)} = 0$*, if any of the following hold for constants c_{10}, c_{11}, c_{12} and ε sufficiently small:*

(a) if $\bar{f}$ satisfies (4.3.1) *and*

$$\kappa/\kappa_0 > c_{10} \max\{(\varepsilon M_0)^{1/4}, (\kappa_f/\kappa_0)^{3/8}\, \mathcal{G}_0^{1/4}\}; \tag{4.4.1}$$

(b) if $\bar{f}$ satisfies (4.3.2) *and*

$$\kappa/\kappa_0 > c_{11} \max\{(\varepsilon M_0)^{1/4}, \mathcal{G}_0^{(2s+5)/(8s+14)}\}; \quad \textit{or} \tag{4.4.2}$$

(c) if $\bar{f}$ satisfies (4.3.3) *and*

$$\kappa/\kappa_0 > c_{12} \max\{(\varepsilon M_0)^{1/4}, F_{\gamma'}(\nu_0^{-1/2}\, \mathcal{G}_0)^{3/8}\, \mathcal{G}_0^{1/4}\}, \tag{4.4.3}$$

where the function $F_{\gamma'}$ is defined in (4.4.41) *below.*

As in Theorem 13, the smallness requirements on ε, which will be given in (4.4.29), (4.4.36) and (4.4.42), are in place purely to simplify the statement of the theorem and can be removed at the expense of longer expressions for the bounds on κ. The function $F_{\gamma'}$ in (4.4.3) is, up to a multiplicative constant, equal to its planar analogue F_γ in (3.4.3).

Proof. This proof essentially follows that of the planar case, with spherical inequalities replacing their planar analogues. We multiply (4.2.10) by $\delta\omega^>$ in L^2 to obtain

$$(\partial_t\delta\omega,\delta\omega^>)+(\partial(\psi^\sharp,\delta\omega),\delta\omega^>)+(\partial(\delta\psi,\omega),\delta\omega^>)+\frac{2}{\varepsilon}(\partial_\phi\delta\psi,\delta\omega^>)=(\mu\Delta\delta\omega,\delta\omega^>). \tag{4.4.4}$$

Harmonic expansion shows that the $2/\varepsilon$ term is 0:

$$\begin{aligned}\frac{2}{\varepsilon}(\partial_\phi\delta\psi,\delta\omega^>) &= \frac{2}{\varepsilon}\sum_{l>\kappa}^{\infty}\sum_{m=-l}^{l} \mathrm{i}m\,\delta\psi_{lm}\overline{\delta\omega^>_{lm}}\\ &= -\frac{2}{\varepsilon}\sum_{l>\kappa}^{\infty}\sum_{m=-l}^{l} \mathrm{i}m\,l(l+1)\,\delta\psi_{lm}\overline{\delta\psi^>_{lm}}\\ &= 0 \qquad\qquad \text{by symmetry,}\end{aligned} \tag{4.4.5}$$

so

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega^>|_2^2+\mu|\nabla\delta\omega^>|_2^2=-(\partial(\psi^\sharp,\delta\omega),\delta\omega^>)-(\partial(\delta\psi^<,\omega),\delta\omega^>)-(\partial(\delta\psi^>,\omega),\delta\omega^>). \tag{4.4.6}$$

For the first term on the right hand side, (4.1.3) and (4.1.4) imply that $(\partial(\psi^\sharp,\delta\omega^>),\delta\omega^>)=0$, so

$$(\partial(\psi^\sharp,\delta\omega),\delta\omega^>)=(\partial(\psi^\sharp,\delta\omega^<),\delta\omega^>). \tag{4.4.7}$$

We split $\omega=\bar\omega+\tilde\omega$ to write the last term of (4.4.6) as

$$(\partial(\delta\psi^>,\omega),\delta\omega^>)=(\partial(\delta\psi^>,\bar\omega),\delta\omega^>)+(\partial(\delta\psi^>,\tilde\omega),\delta\omega^>). \tag{4.4.8}$$

The first term on the right hand side of this becomes

$$\begin{aligned}(\partial(\delta\psi^>,\bar\omega),\delta\omega^>) &= (\partial(\delta\bar\psi^>,\bar\omega),\delta\omega^>)+(\partial(\delta\tilde\psi^>,\bar\omega),\delta\omega^>)\\ &= (\partial(\delta\tilde\psi^>,\bar\omega),\delta\omega^>) && \text{by (4.1.5)}\\ &= (\partial(\delta\tilde\psi^>,\bar\omega),\delta\bar\omega^>)+(\partial(\delta\tilde\psi^>,\bar\omega),\delta\tilde\omega^>)\\ &= (\partial(\bar\omega,\delta\bar\omega^>),\delta\tilde\psi^>)+(\partial(\delta\tilde\psi^>,\bar\omega),\delta\tilde\omega^>) && \text{by (4.1.3)}\\ &= (\partial(\delta\tilde\psi^>,\bar\omega),\delta\tilde\omega^>) && \text{by (4.1.5).}\end{aligned} \tag{4.4.9}$$

In order to apply the bounds we obtained in Lemma 15, we split $\bar\omega=\bar\omega^{<f}+\bar\omega^{>f}$, where $\bar{\omega}^{<f} = \mathsf{P}_{\kappa_f}\bar{\omega}$ and $\bar{\omega}^{>f} = \bar{\omega} - \bar{\omega}^{<f}$. Now

$$(\partial(\delta\tilde{\psi}^{>}, \bar{\omega}), \delta\tilde{\omega}^{>}) = (\partial(\delta\tilde{\psi}^{>}, \bar{\omega}^{<f}), \delta\tilde{\omega}^{>}) + (\partial(\delta\tilde{\psi}^{>}, \bar{\omega}^{>f}), \delta\tilde{\omega}^{>}). \tag{4.4.10}$$

Thus (4.4.6) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega^{>}|^2 + \mu|\nabla\delta\omega^{>}|^2 \\
&\quad= -\,(\partial(\psi^{\sharp}, \delta\omega^{<}), \delta\omega^{>}) - (\partial(\delta\psi^{<}, \omega), \delta\omega^{>}) - (\partial(\delta\psi^{>}, \tilde{\omega}), \delta\omega^{>}) \\
&\qquad - (\partial(\delta\tilde{\psi}^{>}, \bar{\omega}^{<f}), \delta\tilde{\omega}^{>}) - (\partial(\delta\tilde{\psi}^{>}, \bar{\omega}^{>f}), \delta\tilde{\omega}^{>}).
\end{aligned} \tag{4.4.11}$$

We bound the first two terms on the right hand side by

$$\begin{aligned}
|(\partial(\psi^{\sharp}, \delta\omega^{<}), \delta\omega^{>})| &= |(\partial(\delta\omega^{<}, \delta\omega^{>}), \psi^{\sharp})| \\
&\le |\psi^{\sharp}|_4|\nabla\delta\omega^{<}|_4|\nabla\delta\omega^{>}|_2 && \text{by Hölder} \\
&\le \frac{4}{\mu}|\psi^{\sharp}|_4^2|\nabla\delta\omega^{<}|_4^2 + \frac{\mu}{16}|\nabla\delta\omega^{>}|_2^2 && \text{by Young} \\
&\le \frac{c}{\mu}|\psi^{\sharp}||\nabla\psi^{\sharp}||\nabla\delta\omega^{<}||\Delta\delta\omega^{<}| + \frac{\mu}{16}|\nabla\delta\omega^{>}|^2 && \text{by Ladyzhenskaya,}
\end{aligned} \tag{4.4.12}$$

and

$$\begin{aligned}
|(\partial(\delta\psi^{<}, \omega), \delta\omega^{>})| &\le |\nabla\delta\psi^{<}|_4|\nabla\omega|_2|\delta\omega^{>}|_4 \\
&\le \frac{c}{\mu\kappa_0}|\nabla\delta\psi^{<}|_4^2|\nabla\omega|_2^2 + c\mu\kappa_0|\delta\omega^{>}|_4^2 \\
&\le \frac{c}{\mu\kappa_0}|\nabla\delta\psi^{<}||\delta\omega^{<}||\nabla\omega|^2 + \frac{\mu\kappa_0}{16}|\delta\omega^{>}||\nabla\delta\omega^{>}| && \text{by Ladyzhenskaya} \\
&\le \frac{c}{\mu\kappa_0}|\nabla\delta\psi^{<}||\delta\omega^{<}||\nabla\omega|^2 + \frac{\mu}{16}|\nabla\delta\omega^{>}|^2 && \text{by Poincaré.}
\end{aligned} \tag{4.4.13}$$

The third term on the right hand side of (4.4.11) is bounded by

$$\begin{aligned}
|(\partial(\delta\psi^{>}, \tilde{\omega}), \delta\omega^{>})| &\le |\nabla\delta\psi^{>}|_4|\nabla\tilde{\omega}|_2|\delta\omega^{>}|_4 \\
&\le c\,|\nabla\delta\psi^{>}|_2^{1/2}|\nabla\delta\omega^{>}|_2^{1/2}|\delta\omega^{>}|_2|\nabla\tilde{\omega}|_2 && \text{by Ladyzhenskaya} \\
&\le \frac{c}{\kappa}|\nabla\delta\omega^{>}|_2|\delta\omega^{>}|_2|\nabla\tilde{\omega}|_2 && \text{by (4.2.13)} \\
&\le \frac{\mu}{16}|\nabla\delta\omega^{>}|^2 + \frac{c}{\mu\kappa^2}|\nabla\tilde{\omega}|^2|\delta\omega^{>}|^2 && \text{by Young.}
\end{aligned} \tag{4.4.14}$$

The rest of the proof is almost exactly identical to that of Theorem 13; we therefore omit duplicate technical details. The fourth term on the right hand side of (4.4.11) is bounded by

$$|(\partial(\delta\psi^{>}, \bar{\omega}^{<f}), \delta\omega^{>})| \leq \frac{\mu}{16}|\nabla\delta\omega^{>}|^2 + \frac{c\,\kappa_0\kappa_f^3}{\mu\kappa^6}|\nabla\omega|^2|\delta\omega^{>}|^2. \tag{4.4.15}$$

We bound the final term of (4.4.11) as

$$|(\partial(\delta\psi^{>}, \bar{\omega}^{>f}), \delta\omega^{>})| \leq \frac{\mu}{16}|\nabla\delta\omega^{>}|^2 + \frac{c\,\kappa_0}{\mu\kappa^2\kappa_f}|\nabla\bar{\omega}^{>f}|^2|\delta\omega^{>}|^2. \tag{4.4.16}$$

Collating these and rearranging, we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega^{>}|^2 + \mu\,|\nabla\delta\omega^{>}|^2 \leq{}& \frac{c}{\mu}|\psi^{\sharp}||\nabla\psi^{\sharp}||\nabla\delta\omega^{<}||\Delta\delta\omega^{<}| + \frac{c}{\mu\kappa_0}|\nabla\delta\psi^{<}||\delta\omega^{<}||\nabla\omega|^2\\ &+ \frac{c}{\mu\kappa^2}|\nabla\tilde{\omega}|^2|\delta\omega^{>}|^2 + \frac{c\,\kappa_0\kappa_f^3}{\mu\kappa^6}|\nabla\omega|^2|\delta\omega^{>}|^2\\ &+ \frac{c\,\kappa_0}{\mu\kappa^2\kappa_f}|\nabla\bar{\omega}^{>f}|^2|\delta\omega^{>}|^2.\end{aligned} \tag{4.4.17}$$

Applying (4.2.13) on the $|\nabla\delta\omega^{>}|^2$ term on the left hand side and rearranging gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega^{>}|^2 + |\delta\omega^{>}|^2\Big(\mu\kappa^2 - \frac{c}{\mu\kappa^2}|\nabla\tilde{\omega}|^2 - \frac{c\,\kappa_0\kappa_f^3}{\mu\kappa^6}|\nabla\omega|^2 - \frac{c\,\kappa_0}{\mu\kappa^2\kappa_f}|\nabla\bar{\omega}^{>f}|^2\Big)&\\ \leq \frac{c}{\mu}|\psi^{\sharp}||\nabla\psi^{\sharp}||\nabla\delta\omega^{<}||\Delta\delta\omega^{<}| + \frac{c}{\mu\kappa_0}|\nabla\delta\psi^{<}||\delta\omega^{<}||\nabla\omega|^2.&\end{aligned} \tag{4.4.18}$$

We apply Lemma 8, with

$$\begin{aligned}\rho &= \mu\kappa^2 - \frac{c}{\mu\kappa^2}|\nabla\tilde{\omega}|^2 - \frac{c\,\kappa_0\kappa_f^3}{\mu\kappa^6}|\nabla\omega|^2 - \frac{c\,\kappa_0}{\mu\kappa^2\kappa_f}|\nabla\bar{\omega}^{>f}|^2,\\ \sigma &= \frac{c}{\mu}|\psi^{\sharp}||\nabla\psi^{\sharp}||\nabla\delta\omega^{<}||\Delta\delta\omega^{<}| + \frac{c}{\mu\kappa_0}|\nabla\delta\psi^{<}||\delta\omega^{<}||\nabla\omega|^2,\\ \xi &= |\delta\omega^{>}|^2,\end{aligned} \tag{4.4.19}$$

i.e. ρ is the bracket on the left hand side of (4.4.18) and σ is the right hand side. In order to validate that the hypothesis of the lemma concerning σ is met, we quote the following result from [8], which give bounds on the derivatives of the vorticity:

$$|\nabla^m\omega(t)|^2_{L^2(S^2)} + \mu\int_0^t |\nabla^{m+1}\omega|^2_{L^2(S^2)}\mathrm{e}^{\nu_0(\tau-t)}\,\mathrm{d}\tau \leq c(m)\frac{\mathcal{G}_m^2(1+c'(m)\nu_0^2\mathcal{G}_0^2)^m}{(\mu\kappa_0)^{2m-2}} \tag{4.4.20}$$

for all $t \geq T_m(|\boldsymbol{v}(0)|_{L^2(S^2)}, |\nabla^{m-1}f|_{L^2(S^2)}; \mu)$. Again, these bounds themselves are independent of the initial data. Thus the hypothesis concerning σ is met because $|\delta\omega^<(t)| \to 0$ as $t \to \infty$ by construction and $|\nabla\omega|$ is bounded when integrated over time (4.4.20). The hypothesis on ξ holds also due to the regularity of the 2D Navier–Stokes equations.

We therefore need to fulfil the hypothesis on ρ, which would follow from

$$\limsup_{t\to\infty} \int_t^{t+1} \Big(\frac{1}{\mu\kappa^2} |\nabla\tilde{\omega}|^2 + \frac{\kappa_0\kappa_f^3}{\mu\kappa^6} |\nabla\omega|^2 + \frac{\kappa_0}{\mu\kappa^2\kappa_f} |\nabla\bar{\omega}^{>f}|^2 \Big) \,\mathrm{d}\tau < c\mu\kappa^2. \tag{4.4.21}$$

This in turn is implied when all of the following are satisfied:

$$\limsup_{t\to\infty} \int_t^{t+1} |\nabla\tilde{\omega}|^2 \,\mathrm{d}\tau < c\,\nu_0^2 (\kappa/\kappa_0)^4, \tag{4.4.22}$$

$$\limsup_{t\to\infty} \int_t^{t+1} |\nabla\omega|^2 \,\mathrm{d}\tau < c\,\nu_0^2 (\kappa/\kappa_0)^8 (\kappa_0/\kappa_f)^3, \text{ and} \tag{4.4.23}$$

$$\limsup_{t\to\infty} \int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2 \,\mathrm{d}\tau < c\,\nu_0^2 (\kappa/\kappa_0)^4 (\kappa_f/\kappa_0). \tag{4.4.24}$$

As before, (4.2.22) implies that the first condition follows for

$$\kappa/\kappa_0 > c\,(\varepsilon M_0/\nu_0^3)^{1/4}. \tag{4.4.25}$$

By applying Lemma 9 to (4.4.20), the second condition (4.4.23) is implied when

$$c\,\nu_0\,\mathcal{G}_0^2 < \nu_0^2 (\kappa/\kappa_0)^8 (\kappa_0/\kappa_f)^3,$$

or equivalently, for

$$\kappa/\kappa_0 > c\,\nu_0^{-1/8} (\kappa_f/\kappa_0)^{3/8} \mathcal{G}_0^{1/4}. \tag{4.4.26}$$

We first consider the case when $\bar{f}$ satisfies (4.3.1). As in the periodic case (3.4.27), we apply Lemma 9 to (4.3.5), so that

$$\int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2 \,\mathrm{d}\tau \le c\,(\varepsilon M_0)^2/\nu_0^3, \tag{4.4.27}$$

i.e. condition (4.4.24) is met when

$$\kappa/\kappa_0 > c\,(\varepsilon M_0)^{1/2} \nu_0^{-5/4} (\kappa_0/\kappa_f)^{1/4}. \tag{4.4.28}$$

This bound is weaker than that of (4.4.25) when

$$\varepsilon M_0 \le c\,\nu_0^2(\kappa_f/\kappa_0), \tag{4.4.29}$$

which we will assume. Combining (4.4.25), (4.4.26) and (4.4.28) gives (4.4.1).

When $\bar{f}$ instead satisfies (4.3.2), we apply Lemma 9 to (4.3.6) to obtain

$$\int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le c\,(\varepsilon M_0)^2/\nu_0^3 + c\,c_\zeta(s)\nu_0(\kappa_0/\kappa_f)^{2s+1}\,\mathcal{G}_0^2 =: I_1^s \tag{4.4.30}$$

where $1/c_\zeta(s) = (2s+1)\zeta(2s+2)$. Thus (4.4.24) would be satisfied when $I_1^s < c\,\mu^2\kappa^4(\kappa_f/\kappa_0)$. Analogously to what we did to (4.4.21), this is implied by

$$(\kappa/\kappa_0)^4 > c\,(\varepsilon M_0)^2\nu_0^{-5}(\kappa_0/\kappa_f), \qquad \text{and} \tag{4.4.31}$$

$$(\kappa/\kappa_0)^4 > c\,c_\zeta(s)\,\nu_0^{-1}(\kappa_0/\kappa_f)^{2s+2}\,\mathcal{G}_0^2. \tag{4.4.32}$$

Since both of conditions (4.4.26) and (4.4.32) must be met, we equate these bounds to find

$$(\kappa_f/\kappa_0)^{2s+7/2} = c\,c_\zeta(s)\nu_0^{-1/2}\,\mathcal{G}_0, \tag{4.4.33}$$

which fixes κ_f, turning both (4.4.23) and (4.4.32) to

$$\kappa/\kappa_0 > c\,\big(c_\zeta(s)^{3/2}\nu_0^{-(s+5/2)}\mathcal{G}_0^{2s+5}\big)^{1/(8s+14)}. \tag{4.4.34}$$

Using κ_f determined in (4.4.33), (4.4.31) becomes

$$\kappa/\kappa_0 > c_s\,(\varepsilon M_0)^{1/2}\nu_0^{-5/4+1/(16s+28)}\mathcal{G}_0^{-1/(8s+14)}, \tag{4.4.35}$$

where $c_s = c\,c_\zeta(s)^{-1/(8s+14)}$, noting that since we require $s > 5/2$, the exponent of $\mathcal{G}_0$ lies between $-1/34$ and 0, giving a weak dependence. This bound is dominated by that of (4.4.25) when

$$\varepsilon M_0 \le c\,c_s^{-4}\nu_0^{2-1/(4s+7)}\mathcal{G}_0^{2/(4s+7)}. \tag{4.4.36}$$

Assuming this, (4.4.2) follows from (4.4.25) and (4.4.34).

Finally, when $\bar{f}$ satisfies (4.3.3), Lemma 9 and (4.3.7) imply

$$\int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le c\,(\varepsilon M_0)^2/\nu_0^3 + c\,\nu_0\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\,\mathcal{G}_0^2. \tag{4.4.37}$$

As in the previous case, (4.4.24) would be satisfied if the following both hold:

$$(\kappa/\kappa_0)^4 > c\,(\varepsilon M_0)^2 \nu_0^{-5}(\kappa_0/\kappa_f) \qquad \text{and,} \tag{4.4.38}$$

$$(\kappa/\kappa_0)^4 > c\,\nu_0^{-1}\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}(\kappa_0/\kappa_f)\,\mathcal{G}_0^2. \tag{4.4.39}$$

Equating (4.4.26) and (4.4.39) gives

$$(\kappa_f/\kappa_0)^{5/2}\,\mathrm{e}^{2\gamma(\kappa_f/\kappa_0-1)} = c_{\gamma'}\,\nu_0^{-1/2}\,\mathcal{G}_0, \tag{4.4.40}$$

which can be inverted to give

$$\kappa_f/\kappa_0 = F_{\gamma'}(\mathcal{G}_0/\nu_0^{1/2}) \quad \text{where} \quad (F_{\gamma'})^{-1}(y) = y^{5/2}\mathrm{e}^{2\gamma(y-1)}/c_{\gamma'}. \tag{4.4.41}$$

With κ_f thus fixed, (4.4.25) would dominate (4.4.38) when

$$\varepsilon M_0 \le c\,\nu_0^2(\kappa_f/\kappa_0). \tag{4.4.42}$$

Assuming this, (4.4.3) follows from (4.4.25) and (4.4.26).

□

Chapter 5

Determining nodes on the periodic β-plane

In this chapter, we state and prove our theorem concerning the determining nodes on the rotating torus $\mathbb{T}^2 = [0, L] \times [-L/2, L/2]$. We begin by introducing the concept of determining nodes, followed by existing results that have been shown for the general (non-rotating) Navier–Stokes equations. We then prove an auxiliary lemma relating norms of a function to its nodal values, which is of key importance to our theorem. Finally, in Section 5.2, we prove our main result of the chapter, as well as discussing its consequences and comparing to its modes' analogue of Theorem 13.

Although closely related to the determining modes of Chapter 3, the nodes differ in that they are concerned with the fluid's velocity or vorticity in physical space, rather than its wavenumber counterparts in Fourier space. This can make the theory more useful in practice, for example when one takes data from physical experiments or observations, rather than having to Fourier transform them first.

We recall the following from Chapter 3. The vorticity form of the β-plane Navier–Stokes equations is given by

$$\partial_t \omega + \partial(\psi, \omega) + \frac{\kappa_0}{\varepsilon}\, \partial_x \psi = \mu \Delta \omega + f. \tag{5.0.1}$$

Then $\delta\omega = \omega - \omega^\sharp$ satisfies

$$\partial_t \delta\omega + \partial(\psi^\sharp, \delta\omega) + \partial(\delta\psi, \omega) + \frac{\kappa_0}{\varepsilon}\,\partial_x \delta\psi = \mu\Delta\delta\omega, \tag{5.0.2}$$

where ω, $\omega^\sharp$ are solutions to (5.0.1) with the same f and different initial conditions. We also recall the definition of the zonal and non-zonal components of the vorticity:

$$\begin{aligned} \bar{\omega}(y,t) &:= \frac{1}{L}\int_0^L \omega(x,y,t)\,\mathrm{d}x, \qquad\qquad \text{and} \\ \tilde{\omega}(x,y,t) &:= \omega(x,y,t) - \bar{\omega}(y,t). \end{aligned}$$

As in previous chapters, where there is no ambiguity, we write $|\cdot| = |\cdot|_{L^2}$, $|\cdot|_p = |\cdot|_{L^p}$ and $(\cdot,\cdot) = (\cdot,\cdot)_{L^2}$.

5.1 Theory of determining nodes

We explain the concept of determining nodes, introduced by and the existence of which was proved by Foias and Temam [15], which is related to the modes presented in Chapter 3. The set $\mathcal{E} = \{\boldsymbol{x}^1, \cdots, \boldsymbol{x}^N\} \subset \mathbb{T}^2$ is said to be a set of determining nodes if

$$\lim_{t\to\infty} \delta\boldsymbol{v}(\boldsymbol{x}^i, t) = 0 \;\text{ for all } i \in \{1,\cdots,N\} \qquad \text{implies} \qquad \lim_{t\to\infty} |\delta\omega(t)|_{L^2(\mathbb{T}^2)} = 0.$$

Foias and Temam's approach to this idea involved bounding the maximal distance between neighbouring nodal points, in order to quantify how "dense" the points would have to be within the domain. Slightly more recently, Jones and Titi [12] took a different approach and derived bounds on the number of nodal points required for the general (1.0.1) case:

Theorem 17 (Jones and Titi '93)**.** *Let $\boldsymbol{v}$ and $\boldsymbol{v}^\#$ satisfy* (1.0.7)*. There exists an absolute constant c_{13} and a set of determining nodes $\mathcal{E} = \{\boldsymbol{x}^1, \cdots, \boldsymbol{x}^N\} \subset \mathbb{T}^2$, where*

$$N \ge c_{13}\,\mathcal{G}_0,$$

i.e. $\lim_{t\to\infty} |\boldsymbol{v}(\boldsymbol{x}^i,t) - \boldsymbol{v}^\#(\boldsymbol{x}^i,t)| = 0$ *for* $i \in \{1,\cdots,N\}$ *implies* $\lim_{t\to\infty} |\delta\omega(t)|_{L^2(\mathbb{T}^2)} =$

0.

It is believed that this bound is qualitatively optimal, in the sense that it agrees with estimates based on physical principles conjectured by Manley and Trève [14]. We will prove improved bounds on the number of nodes, under the additional assumption that the domain is undergoing a differential rotation.

We also require the following estimates from [12], which relate the H^s and L^∞ norms of a function to its value at the nodes.

Lemma 18. *Let $u \in H^2(\mathbb{T}^2)$. Define*

$$\eta(u) := \max_{1\le i\le N} |u(\boldsymbol{x}^i)|,$$

where $\mathbb{T}^2$ is divided into N equal squares with corners at $\boldsymbol{x}^i$, for $i \in \{1, \cdots, N\}$. Then

$$|u|^2_{L^2(\mathbb{T}^2)} \le c_{14}\Big(L^2\eta^2(u) + \frac{L^4}{N^2}|\Delta u|^2_{L^2(\mathbb{T}^2)}\Big), \tag{5.1.1}$$

$$|\nabla u|^2_{L^2(\mathbb{T}^2)}, |u|^2_{L^\infty(\mathbb{T}^2)} \le c_{14}\Big(N\eta^2(u) + \frac{L^2}{N}|\Delta u|^2_{L^2(\mathbb{T}^2)}\Big), \tag{5.1.2}$$

where c_{14} is an absolute constant.

We will give the proof below, written in a slightly different manner (and possibly different constants) to that in [12], so that it will be easier to compare with the analogous collocation lemma for the sphere, presented later in Chapter 6.

Proof. Let $Q = [0, l] \times [0, l]$ be a square and assume that we know the values of $v \in H^2(Q)$ at the corners $(0, 0)$, $(0, l)$, $(l, 0)$ and (l, l). We will obtain bounds on the "one-dimensional" L^2 norms of v over Q (i.e. norms with one variable fixed), then use these to bound $|u|_{L^2(\mathbb{T}^2)}$.

We first aim to integrate $v \in H^2(Q)$ from $(0, 0)$ to $\boldsymbol{P} = (x_p, y_p) \in R := [l/2, l]\times[l/2, l]$ (i.e. R is the quadrant of Q furthest from $(0, 0)$). By the fundamental theorem of calculus,

$$v^2(x_p, y) = v^2(x, y) + \int_x^{x_p} \partial_x v^2(x', y)\,\mathrm{d}x', \tag{5.1.3}$$

where $x \in [0, x_p]$. Integrating this with respect to x and y over the rectangle $\Omega_p := [0, x_p] \times [0, y_p]$ gives

$$\int_0^{y_p} \int_0^{x_p} v^2(x_p, y)\,\mathrm{d}x\,\mathrm{d}y = |v|^2_{L^2(\Omega_p)} + \int_0^{y_p} \int_0^{x_p} \int_x^{x_p} \partial_x v^2(x', y)\,\mathrm{d}x'\,\mathrm{d}x\,\mathrm{d}y. \tag{5.1.4}$$

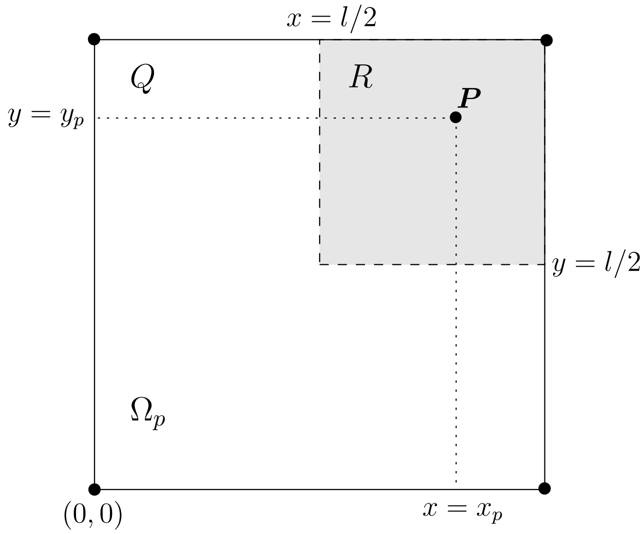

Figure 5.1: Illustration of R and Ω_p within Q.

The left hand side of (5.1.4) is equal to

$$\int_0^{y_p} \int_0^{x_p} v^2(x_p, y)\,\mathrm{d}x\,\mathrm{d}y = |x_p| \int_0^{y_p} v^2(x_p, y)\,\mathrm{d}y = |x_p| |v(x_p, \cdot)|^2_{L^2(0, y_p)}, \tag{5.1.5}$$

where we have used the absolute value of x_p to emphasize that this is the length $(x_p - 0)$, rather than the coordinate itself. The right hand side of (5.1.4) becomes

$$\begin{aligned}
&|v|^2_{L^2(\Omega_p)} + \int_0^{y_p} \int_0^{x_p} \int_x^{x_p} \partial_x v^2(x', y)\,\mathrm{d}x'\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \leq |v|^2_{L^2(\Omega_p)} + 2 \int_0^{y_p} \int_0^{x_p} \int_x^{x_p} |v\,\partial_x v|(x', y)\,\mathrm{d}x'\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad \leq |v|^2_{L^2(\Omega_p)} + 2 \int_0^{y_p} \int_0^{x_p} \int_0^{x_p} |v\,\partial_x v|(x', y)\,\mathrm{d}x'\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = |v|^2_{L^2(\Omega_p)} + 2|x_p| |v\,\partial_x v|_{L^1(\Omega_p)} \\
&\qquad \leq |v|^2_{L^2(\Omega_p)} + 2|x_p| |v|_{L^2(\Omega_p)} |\partial_x v|_{L^2(\Omega_p)} && \text{by Hölder} \\
&\qquad \leq 2|v|^2_{L^2(\Omega_p)} + |x_p|^2 |\partial_x v|^2_{L^2(\Omega_p)} && \text{by Young.}
\end{aligned} \tag{5.1.6}$$

Thus applying (5.1.5) and (5.1.6) to (5.1.4) gives

$$|v(x_p,\cdot)|^2_{L^2(0,y_p)} \le \frac{2}{|x_p|}|v|^2_{L^2(\Omega_p)} + |x_p||\partial_x v|^2_{L^2(\Omega_p)}. \tag{5.1.7}$$

Similarly to (5.1.3) but by integrating in the y-direction instead, we have

$$v^2(x,y_p) = v^2(x,y) + \int_y^{y_p} \partial_y v^2(x,y')\,\mathrm{d}y, \tag{5.1.8}$$

where $y \in [0,y_p]$. By symmetry, we obtain

$$|v(\cdot,y_p)|^2_{L^2(0,x_p)} \le \frac{2}{|y_p|}|v|^2_{L^2(\Omega_p)} + |y_p||\partial_y v|^2_{L^2(\Omega_p)}. \tag{5.1.9}$$

Equipped with (5.1.7) and (5.1.9), we integrate u from $(0,0)$ to $(x,y) \in R$ as follows. By the triangle inequality, we have

$$u(x,y) - u(0,0) \le |u(x,y) - u(0,0)| \le |u(x,y) - u(x,0)| + |u(x,0) - u(0,0)|,$$

which, after rearranging, implies that

$$u^2(x,y) \le 3u^2(0,0) + 3|u(x,y) - u(x,0)|^2 + 3|u(x,0) - u(0,0)|^2. \tag{5.1.10}$$

By applying (5.1.7) to $v = \partial_y u$, x_p replaced by x and y_p by y, the second term on the right hand side of this is bounded by

$$\begin{aligned}
|u(x,y) - u(x,0)|^2 &= \left|\int_0^y \partial_y u(x,y')\mathrm{d}y'\right|^2 \le |\partial_y u(x,\cdot)|^2_{L^1(0,y)} \\
&\le |1|^2_{L^2(0,y)}|\partial_y u(x,\cdot)|^2_{L^2(0,y)} && \text{by Hölder} \\
&= |y||\partial_y u(x,\cdot)|^2_{L^2(0,y)} \\
&\le |l|\Big(\frac{2}{|x|}|\partial_y u|^2_{L^2([0,x]\times[0,y])} + |x||\partial^2_{xy} u|^2_{L^2([0,x]\times[0,y])}\Big) && \text{by (5.1.7)} \\
&\le |l|\Big(\frac{2}{|l/2|}|\partial_y u|^2_{L^2(Q)} + |l||\partial^2_{xy} u|^2_{L^2(Q)}\Big) \\
&= 4|\partial_y u|^2_{L^2(Q)} + |l|^2|\partial^2_{xy} u|^2_{L^2(Q)}.
\end{aligned} \tag{5.1.11}$$

Similarly, we would like to apply (5.1.9) to $v = \partial_x u$. We note that we need to rotate Q in order to do so, since we require $y_p = 0$.

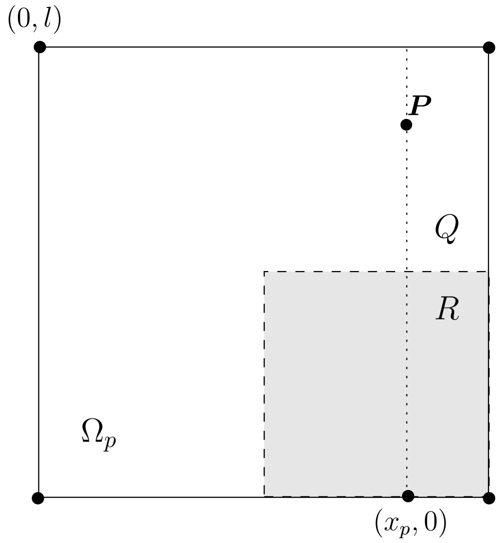

Figure 5.2: Rotation of Q in order to define $R \ni (x_p, 0)$ and corresponding Ω_p.

Thus by integrating $v = \partial_x u$ from $(0, l)$ instead of $(0, 0)$, we bound the last term of (5.1.10) by

$$
\begin{aligned}
|u(x,0)-u(0,0)|^2 &= \left|\int_0^x \partial_x u(x',0)\,\mathrm{d}x'\right|^2 \le |\partial_x u(\cdot,0)|^2_{L^1(0,x)} \\
&\le |1|^2_{L^2(0,x)} |\partial_x u(\cdot,0)|^2_{L^2(0,x)} && \text{by Hölder} \\
&= |l||\partial_x u(\cdot,0)|^2_{L^2(0,x)} \\
&\le |l|\Big(\frac{2}{|l/2|}|\partial_x u|^2_{L^2(Q)} + |l||\partial^2_{xy} u|^2_{L^2(Q)}\Big) && \text{by (5.1.9)} \\
&= 4|\partial_x u|^2_{L^2(Q)} + |l|^2 |\partial^2_{xy} u|^2_{L^2(Q)}. && (5.1.12)
\end{aligned}
$$

Applying (5.1.11) and (5.1.12) to (5.1.10) gives

$$
\begin{aligned}
u^2(x,y) &\le 3u^2(0,0) + 3|u(x,y) - u(x,0)|^2 + 3|u(x,0) - u(0,0)|^2 \\
&\le 3u^2(0,0) + 12|\partial_y u|^2_{L^2(Q)} + 12|\partial_x u|^2_{L^2(Q)} + 6|l|^2|\partial^2_{xy} u|^2_{L^2(Q)} \\
&\le 3u^2(0,0) + 24|\nabla u|^2_{L^2(Q)} + 6|l|^2|\partial^2_{xy} u|^2_{L^2(Q)} \\
&= 3u^2(0,0) + 24|\nabla u|^2_{L^2(Q)} + 6|Q||\partial^2_{xy} u|^2_{L^2(Q)}. \qquad (5.1.13)
\end{aligned}
$$

We integrate this over R, giving

$$
|u|^2_{L^2(R)} \le 3|R|u^2(0,0) + 24|R||\nabla u|^2_{L^2(Q)} + 6|R||Q||\partial^2_{xy} u|^2_{L^2(Q)}. \qquad (5.1.14)
$$

We then "rotate" R three times and take the sum to cover Q (i.e. we integrate from the three other nodes too, over all the quadrants):

$$\begin{aligned}|u|^2_{L^2(Q)} \leq 3|Q| \max\{u^2(0,0), u^2(0,l), u^2(l,0), u^2(l,l)\} \\ + 24|Q||\nabla u|^2_{L^2(Q)} + 6|Q|^2|\partial^2_{xy}u|^2_{L^2(Q)}.\end{aligned} \tag{5.1.15}$$

Now that we have bounds on $|u|_{L^2(Q)}$, we use these to bound $|u|_{L^2(\mathbb{T}^2)}$. We divide $\mathbb{T}^2$ into N equal squares Q_i, $i \in \{1, \cdots, N\}$, each of side length $L/\sqrt{N}$, and place the nodes at their corners (noting that periodicity implies that we have $\sqrt{N}$ rows and columns of nodes, rather than $\sqrt{N}+1$). We apply (5.1.15) to $Q = Q_i$, which gives

$$\begin{aligned}|u|^2_{L^2(Q_i)} &\leq 3|Q_i|\eta^2(u) + 24|Q_i||\nabla u|^2_{L^2(Q_i)} + 6|Q_i|^2|\partial^2_{xy}u|^2_{L^2(Q_i)} \\ &= 3|Q_i|\eta^2(u) + 24\frac{L^2}{N}|\nabla u|^2_{L^2(Q_i)} + 6\frac{L^4}{N^2}|\partial^2_{xy}u|^2_{L^2(Q_i)}.\end{aligned} \tag{5.1.16}$$

Taking the sum over $i \in \{1, \cdots, N\}$ gives

$$\begin{aligned}|u|^2_{L^2(\mathbb{T}^2)} &\leq 3|\mathbb{T}^2|\eta^2(u) + 24\frac{L^2}{N}|\nabla u|^2_{L^2(\mathbb{T}^2)} + 6\frac{L^4}{N^2}|\partial^2_{xy}u|^2_{L^2(\mathbb{T}^2)} \\ &\leq 3|\mathbb{T}^2|\eta^2(u) + 24\frac{L^2}{N}|\nabla u|^2_{L^2(\mathbb{T}^2)} + 3\frac{L^4}{N^2}|\Delta u|^2_{L^2(\mathbb{T}^2)} \\ &= 3L^2\eta^2(u) - 24\frac{L^2}{N}(u, \Delta u)_{L^2(\mathbb{T}^2)} + 3\frac{L^4}{N^2}|\Delta u|^2_{L^2(\mathbb{T}^2)} && \text{by (3.3.11)} \\ &\leq 3L^2\eta^2(u) + 24\frac{L^2}{N}|u\Delta u|_{L^1(\mathbb{T}^2)} + 3\frac{L^4}{N^2}|\Delta u|^2_{L^2(\mathbb{T}^2)} \\ &\leq 3L^2\eta^2(u) + 24\frac{L^2}{N}|u|_{L^2(\mathbb{T}^2)}|\Delta u|_{L^2(\mathbb{T}^2)} + 3\frac{L^4}{N^2}|\Delta u|^2_{L^2(\mathbb{T}^2)} && \text{by Hölder,}\end{aligned} \tag{5.1.17}$$

where the second line is due to integration by parts and the periodicity of the domain:

$$\begin{aligned}|\partial^2_{xy}u|^2 &= \int_{-L/2}^{L/2}\int_0^L (\partial^2_{xy}u)^2 \,\mathrm{d}\boldsymbol{x} \\ &= \int_{-L/2}^{L/2}\left(\Big[\partial_y u\, \partial^2_{xy}u\Big]_{x=0}^{x=L} - \int_0^L \partial_y u\, \partial^3_{xxy}u \,\mathrm{d}x\right)\mathrm{d}y \\ &= -\int_{-L/2}^{L/2}\int_0^L \partial_y u\, \partial^3_{xxy}u \,\mathrm{d}\boldsymbol{x} \\ &= -\int_0^L\left(\Big[\partial_y u\, \partial^2_{xx}u\Big]_{y=-L/2}^{y=L/2} - \int_{-L/2}^{L/2} \partial^2_{yy}u\, \partial^2_{xx}u \,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^L\int_{-L/2}^{L/2} \partial^2_{xx}u\, \partial^2_{yy}u \,\mathrm{d}\boldsymbol{x}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\int_{\mathbb{T}^2}\Big((\partial^2_{xx}u)^2 + 2\partial^2_{xx}u\,\partial^2_{yy}u + (\partial^2_{yy}u)^2\Big)\,\mathrm{d}\boldsymbol{x}\\
&= \frac{1}{2}|\Delta u|^2.
\end{aligned}
$$

Thus (5.1.17) becomes

$$
\begin{aligned}
|u|^2 &\le 3L^2\eta^2(u) + 24\frac{L^2}{N}|u||\Delta u| + 3\frac{L^4}{N^2}|\Delta u|^2\\
&\le 3L^2\eta^2(u) + \frac{1}{2}|u|^2 + 288\frac{L^4}{N^2}|\Delta u|^2 + 3\frac{L^4}{N^2}|\Delta u|^2 \qquad \text{by Young,}
\end{aligned}
$$

leading to

$$
|u|^2 \le 6L^2\eta^2(u) + 582\frac{L^4}{N^2}|\Delta u|^2,
$$

which is (5.1.1). Applying this to (3.3.11) gives the first part of (5.1.2):

$$
\begin{aligned}
|\nabla u|^2 &= -(u,\Delta u) \le |u\Delta u|_1 \le |u||\Delta u| && \text{by Hölder}\\
&\le \frac{N}{2L^2}|u|^2 + \frac{L^2}{2N}|\Delta u|^2 && \text{by Young}\\
&\le \frac{N}{2L^2}\Big(6L^2\eta^2(u) + 582\frac{L^4}{N^2}|\Delta u|^2\Big) + \frac{L^2}{2N}|\Delta u|^2 && \text{by (5.1.1)}\\
&= 3N\eta^2(u) + \frac{583}{2}\frac{L^2}{N}|\Delta u|^2.
\end{aligned}
$$

Using Agmon's inequality instead of (3.3.11) and following the same steps as above gives

$$
\begin{aligned}
|u|^2_\infty &\le c\,|u||\Delta u| && \text{by Agmon}\\
&\le \cdots \le c\,N\eta^2(u) + c\,\frac{L^2}{N}|\Delta u|^2,
\end{aligned}
$$

which is the second part of (5.1.2). This concludes the proof. □

An alternative approach to prove the same lemma is to adapt the following lemma from Pasciak [23] (see [24] also). The collocation operator $\mathbb{I}_N$ describes the unique operator that interpolates u to the equally spaced nodes of $\mathbb{T}^2$, via the approximation space $S_N := \operatorname{span}\{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} : |\boldsymbol{k}| \le 2\pi N/L\}$. It can be compared to the inverse of the discrete Fourier transform. Our N below refers to the total number of nodal points (as is in the entirety of the thesis), which corresponds to N^2 in Pasciak's paper.

Lemma 19. *Let $\mathbb{I}_N$ denote the collocation operator on $u \in L^2(\mathbb{T}^2)$. Then there exists an absolute constant c_{15} such that*

$$|u - \mathbb{I}_N u|_{L^2(\mathbb{T}^2)} \leq c_{15} \frac{|\mathbb{T}^2|}{N} |\Delta u|_{L^2(\mathbb{T}^2)}. \tag{5.1.18}$$

The lemma immediately implies that

$$\int_{\mathbb{T}^2} \Big(u(\boldsymbol{x}) - \mathbb{I}_N u(\boldsymbol{x})\Big)^2 \,\mathrm{d}\boldsymbol{x} = |u - \mathbb{I}_N u|^2 \leq c \frac{|\mathbb{T}^2|^2}{N^2} |\Delta u|^2.$$

Expanding the brackets and rearranging gives

$$\begin{aligned}
\int_{\mathbb{T}^2} \left(u^2(\boldsymbol{x}) + \Big(\mathbb{I}_N u(\boldsymbol{x})\Big)^2\right) \mathrm{d}\boldsymbol{x} &\leq 2 \int_{\mathbb{T}^2} |u(\boldsymbol{x}) \mathbb{I}_N u(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x} + c \frac{|\mathbb{T}^2|^2}{N^2} |\Delta u|^2 \\
&\leq 2|u||\mathbb{I}_N u| + c \frac{|\mathbb{T}^2|^2}{N^2} |\Delta u|^2 \qquad \text{by Hölder} \\
&\leq \frac{1}{2}|u|^2 + 2|\mathbb{I}_N u|^2 + c \frac{|\mathbb{T}^2|^2}{N^2} |\Delta u|^2 \qquad \text{by Young.}
\end{aligned}$$

Removing the second term on the left hand side and rearranging again gives (see [23] for the last inequality)

$$|u|^2 \leq 4\,|\mathbb{I}_N u|^2 + c \frac{|\mathbb{T}^2|^2}{N^2} |\Delta u|^2 \leq c\,|\mathbb{T}^2| \eta^2(u) + c \frac{|\mathbb{T}^2|^2}{N^2} |\Delta u|^2,$$

which is exactly (5.1.1) up to a constant. As in our proof, (5.1.2) can be easily derived from this.

5.2 Bounds on the number of determining nodes

As with our proof on the determining modes (Theorem 13), we consider different forms of the zonal forcing $\bar{f}$, as given in (3.3.1) to (3.3.3), and the consequences they have on the number of determining nodes.

Theorem 20 (Determining nodes on the periodic β-plane)**.** *Let $\delta\omega$ be the solution of* (5.0.2) *with $f \in H^2(\mathbb{T}^2)$. Then there exists a set of determining nodes $\mathcal{E} = \{\boldsymbol{x}^1, \cdots, \boldsymbol{x}^N\} \subset \mathbb{T}^2$, when* *(a)* $\bar{f}$ *satisfies* (3.3.1) *and*

$$N > c_{16}(\nu_0)\ \max\Big\{\varepsilon^{1/2} M_0^{1/2}, (\kappa_f/\kappa_0)^{1/3}\, \mathcal{G}_0^{2/3}\Big\},\ \ or \tag{5.2.1}$$

(b) $\bar{f}$ *satisfies* (3.3.2) *and*

$$N > c_{17}(\nu_0, s)\ \max\Big\{\varepsilon^{1/2} M_0^{1/2},\ \mathcal{G}_0^{(4s+5)/(6s+5)}\Big\}\ \ or \tag{5.2.2}$$

(c) $\bar{f}$ *satisfies* (3.3.3) *and*

$$N > c_{18}(\nu_0)\ \max\Big\{\varepsilon^{1/2} M_0^{1/2},\ F_\gamma^*(\nu_0^{-1}\, \mathcal{G}_0^{2/3})^{1/3}\, \mathcal{G}_0^{2/3}\Big\}, \tag{5.2.3}$$

for constants c_{16}*,* c_{17}*,* c_{18}*,* F_γ^* *defined as in* (5.2.27) *below and sufficiently small* ε*.*

The above-mentioned smallness requirements on ε are given in (5.2.18), (5.2.24) and (5.2.28). We have chosen to state them later and separately from the statement of the theorem for clarity; one could include them here in exchange for longer and more complicated-looking bounds on N.

We note that since we will use Lemma 18 in the proof that follows, the implicit assumption of the theorem is that N is the square of an integer.

Similarly to Theorem 13, (5.2.27) implies that $F_\gamma^*(u)$ scales as $\log u/(2\gamma)$ for large u, so that the second and larger (dominant) bound in (5.2.3) scales basically as $\mathcal{G}_0^{2/3} \log \mathcal{G}_0$.

We also note immediately that these bounds are qualitatively worse than the modes' equivalent given in Theorem 13, since one would expect N to scale as $(\kappa/\kappa_0)^2$ (this is apparent by comparing the dimensions of N and κ/κ_0), but our bounds on N are worse. Our speculation is that this is purely due to a technical and artificial issue such that we cannot (almost entirely) carry over the proof for the determining modes, rather than there being a fundamental qualitative difference between the modes and nodes.

Proof. We begin by multiplying (5.0.2) by $\delta\omega$ in L^2:

$$(\partial_t\delta\omega,\delta\omega)+(\partial(\psi^\sharp,\delta\omega),\delta\omega)+(\partial(\delta\psi,\omega),\delta\omega)+\frac{\kappa_0}{\varepsilon}(\partial_x\delta\psi,\delta\omega)=\mu(\Delta\delta\omega,\delta\omega). \quad (5.2.4)$$

As in (3.4.5), the fourth term is 0:

$$\begin{aligned}\frac{\kappa_0}{\varepsilon}(\partial_x\delta\psi,\delta\omega) &= \frac{\kappa_0}{\varepsilon}\sum_{\boldsymbol{k}} \mathrm{i}k_1\delta\psi_{\boldsymbol{k}}\overline{\delta\omega_{\boldsymbol{k}}}\\ &= \frac{\kappa_0}{\varepsilon}\sum_{\boldsymbol{k}} -\mathrm{i}k_1|\boldsymbol{k}|^2\delta\psi_{\boldsymbol{k}}\overline{\delta\psi_{\boldsymbol{k}}}=0.\end{aligned}$$

Furthermore, the second term is also 0, due to (3.1.17). Thus (5.2.4) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega|^2+(\partial(\delta\psi,\omega),\delta\omega)=\mu(\Delta\delta\omega,\delta\omega), \quad (5.2.5)$$

of which the right hand side becomes, as in (3.3.12),

$$\mu(\Delta\delta\omega,\delta\omega)=-\mu|\nabla\delta\omega|^2. \quad (5.2.6)$$

By splitting the vorticity into $\omega=\bar\omega+\tilde\omega$, the second term of (5.2.5) becomes

$$(\partial(\delta\psi,\omega),\delta\omega)=(\partial(\delta\psi,\bar\omega),\delta\omega)+(\partial(\delta\psi,\tilde\omega),\delta\omega). \quad (5.2.7)$$

Similarly to (3.4.10), we further split the zonal vorticity $\bar\omega$ into $\bar\omega=\bar\omega^{<f}+\bar\omega^{>f}$, where $\bar\omega^{<f}=\mathsf{P}_{\kappa_f}\bar\omega$ and $\bar\omega^{>f}=\bar\omega-\bar\omega^{<f}$, for some $\kappa_f\geq\kappa_0$ that we will fix later. Rearranging (5.2.5) thus gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega|^2+\mu|\nabla\delta\omega|^2=-(\partial(\delta\psi,\tilde\omega),\delta\omega)-(\partial(\delta\psi,\bar\omega^{<f}),\delta\omega)-(\partial(\delta\psi,\bar\omega^{>f}),\delta\omega). \quad (5.2.8)$$

Now we choose $\mathcal{E}$ to be the set of N equally spaced points over $\mathbb{T}^2$ (i.e. arranged in a square grid). We use (5.1.2) to bound the first term on the right hand side of (5.2.8):

$$\begin{aligned}|(\partial(\delta\psi,\tilde\omega),\delta\omega)| &\leq |\nabla\delta\psi|_\infty|\nabla\tilde\omega|_2|\delta\omega|_2 && \text{by Hölder}\\ &\leq c\mu\frac{N}{L^2}|\nabla\delta\psi|_\infty^2+\frac{cL^2}{\mu N}|\nabla\tilde\omega|^2|\delta\omega|^2 && \text{by Young}\\ &\leq c\mu\frac{N}{L^2}\Big[N\eta^2(\nabla\delta\psi)+\frac{L^2}{N}|\nabla\delta\omega|^2\Big]+\frac{cL^2}{\mu N}|\nabla\tilde\omega|^2|\delta\omega|^2 && \text{by (5.1.2).} \quad (5.2.9)\end{aligned}$$

We then use (5.1.1) on the second term on the right hand side of (5.2.8):

$$\begin{aligned}
|(\partial(\delta\psi,\bar\omega^{<f}),\delta\omega)| &\le |\nabla\bar\omega^{<f}|_\infty|\nabla\delta\psi|_2|\delta\omega|_2 \\
&\le c\,\kappa_0^{1/2}|\nabla\bar\omega^{<f}|^{1/2}|\Delta\bar\omega^{<f}|^{1/2}|\nabla\delta\psi||\delta\omega| && \text{by Agmon} \\
&\le c\,(\kappa_0\kappa_f)^{1/2}|\nabla\bar\omega^{<f}||\nabla\delta\psi||\delta\omega| && \text{by (3.1.33)} \\
&\le c\,(\kappa_0\kappa_f)^{1/2}|\nabla\omega||\nabla\delta\psi||\delta\omega| \\
&\le c\mu\frac{N}{L^4}|\nabla\delta\psi|^2+\frac{cL^4}{\mu N^2}\kappa_0\kappa_f|\nabla\omega|^2|\delta\omega|^2 && \text{by Young} \\
&\le c\mu\frac{N^2}{L^4}\Big[L^2\eta^2(\nabla\delta\psi)+\frac{L^4}{N^2}|\nabla\delta\omega|^2\Big]+\frac{cL^4}{\mu N^2}\kappa_0\kappa_f|\nabla\omega|^2|\delta\omega|^2 && \text{by (5.1.1).}
\end{aligned} \tag{5.2.10}$$

We bound the last term of (5.2.8) as

$$\begin{aligned}
|(\partial(\delta\psi,\bar\omega^{>f}),\delta\omega)| &\le |\nabla\delta\psi|_\infty|\nabla\bar\omega^{>f}|_2|\delta\omega|_2 \\
&\le c\mu\frac{N}{L^2}|\nabla\delta\psi|_\infty^2+\frac{cL^2}{\mu N}|\nabla\bar\omega^{>f}|_2^2|\delta\omega|_2^2 && \text{by Young} \\
&\le c\mu\frac{N}{L^2}\Big[N\eta^2(\nabla\delta\psi)+\frac{L^2}{N}|\nabla\delta\omega|^2\Big]+\frac{cL^2}{\mu N}|\nabla\bar\omega^{>f}|^2|\delta\omega|^2 && \text{by (5.1.2).}
\end{aligned} \tag{5.2.11}$$

Finally, we also apply (5.1.2) to the second term on the left hand side of (5.2.8) and rearrange to obtain

$$\begin{aligned}
|\delta\omega|^2 &\le c\left(N\eta^2(\nabla\delta\psi)+\frac{L^2}{N}|\nabla\delta\omega|^2\right) \\
\Rightarrow c\,\mu\frac{N}{L^2}|\delta\omega|^2 &- c\mu\frac{N^2}{L^2}\,\eta^2(\nabla\delta\psi)\le\mu|\nabla\delta\omega|^2.
\end{aligned} \tag{5.2.12}$$

Thus putting together (5.2.8) to (5.2.12) gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega|^2+|\delta\omega|^2\Big[c\mu\frac{N}{L^2}-\frac{cL^2}{\mu N}|\nabla\tilde\omega|^2-\frac{cL^4}{\mu N^2}\kappa_0\kappa_f|\nabla\omega|^2-\frac{cL^2}{\mu N}|\nabla\bar\omega^{>f}|^2\Big]& \\
\le c\mu\frac{N^2}{L^2}\,\eta^2(\nabla\delta\psi).&
\end{aligned} \tag{5.2.13}$$

We seek to apply Lemma 8 to $\xi=|\delta\omega|^2$, ρ being the bracket on the left hand side and σ the right hand side of (5.2.13). The hypothesis of the lemma on $\sigma=(c\mu N/L^2)\,\eta^2(\nabla\delta\psi)$ is met because $\nabla\delta\psi(\boldsymbol{x}^i,t)\to 0$ as $t\to\infty$ for all i and $|\nabla\omega|$ is bounded due to (3.4.19), while that on ξ follows from the regularity of the Navier–Stokes equations.

The hypothesis on ρ would follow immediately if

$$\limsup_{t\to\infty}\int_t^{t+1}\Big(\frac{L^2}{\mu N}|\nabla\tilde{\omega}|^2+\frac{L^4}{\mu N^2}\kappa_0\kappa_f|\nabla\omega|^2+\frac{L^2}{\mu N}|\nabla\bar{\omega}^{>f}|^2\Big)\,\mathrm{d}\tau<c\mu\frac{N}{L^2},$$

which is equivalent to

$$\limsup_{t\to\infty}\int_t^{t+1}\Big(\frac{1}{\nu_0 N}|\nabla\tilde{\omega}|^2+\frac{1}{\nu_0 N^2}\frac{\kappa_f}{\kappa_0}|\nabla\omega|^2+\frac{1}{\nu_0 N}|\nabla\bar{\omega}^{>f}|^2\Big)\,\mathrm{d}\tau<c\,\nu_0 N, \qquad (5.2.14)$$

where we recall $\nu_0=\mu\kappa_0^2=4\pi^2\mu/L^2$. Without any further assumptions, we would require that each of the three terms on the left hand side satisfies the inequality independently.

For the first term, we note that (3.2.9) implies

$$\int_t^{t+1}|\nabla\tilde{\omega}|^2\,\mathrm{d}\tau\le\varepsilon M_0/\nu_0,$$

so (5.2.14) for the $|\nabla\tilde{\omega}|^2$ term would be satisfied for

$$N^2>c\,\varepsilon M_0/\nu_0^3. \qquad (5.2.15)$$

For the second term, we recall from Chapter 3 that (3.4.19) implies

$$\int_t^{t+1}|\nabla\omega|^2\,\mathrm{d}\tau\le c\,\nu_0\mathcal{G}_0^2,$$

so the $|\nabla\omega|$ part of (5.2.14) is implied by

$$N>\frac{c}{\nu_0^{1/3}}\Big(\frac{\kappa_f}{\kappa_0}\Big)^{1/3}\mathcal{G}_0^{2/3}. \qquad (5.2.16)$$

For the inequality involving $|\nabla\bar{\omega}^{>f}|^2$ in (5.2.14), we need to handle the cases separately according to the different forms of $\bar{f}$.

We consider first when $\bar{f}$ satisfies (3.3.1). By (3.3.4),

$$\int_t^{t+1}|\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau\le c\,\varepsilon^2M_0^2/\nu_0^3,$$

so the $|\nabla\bar{\omega}^{>f}|$ part of (5.2.14) holds if

$$N > c\,\varepsilon M_0/\nu_0^{5/2}. \tag{5.2.17}$$

This bound is dominated by (5.2.15) when

$$\varepsilon M_0 \le c\,\nu_0^2. \tag{5.2.18}$$

Assuming this, (5.2.1) follows from (5.2.15) and (5.2.16).

For $\bar{f}$ instead satisfying (3.3.2), we apply Lemma 9 to (3.3.5) to remark that

$$\int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2\,\mathrm{d}\tau \le c\,\varepsilon^2 M_0^2/\nu_0^3 + c\,c_\zeta(s)\,\nu_0\,(\kappa_0/\kappa_f)^{2s+1}\,\mathcal{G}_0^2 = I_1, \tag{5.2.19}$$

where $1/c_\zeta(s) = (2s+1)\,\zeta(2s+2)$. Therefore, the $|\nabla\bar{\omega}^{>f}|^2$ part of (5.2.14) would be satisfied if $I_1 \le c\,\nu_0^2 N^2$; analogously to what we did with (5.2.14), this in turn is implied by

$$N^2 > c\,(\varepsilon M_0)^2/\nu_0^5, \qquad \text{and} \tag{5.2.20}$$

$$N^2 > \frac{c}{\nu_0}\,c_\zeta(s)(\kappa_0/\kappa_f)^{2s+1}\,\mathcal{G}_0^2. \tag{5.2.21}$$

Since (5.2.16) and (5.2.21) must both hold, we equate these bounds (noting that (5.2.21) scales as N^2 instead of N) to find the optimal κ_f that will minimise the bounds on N:

$$\begin{aligned} c\,\frac{c_\zeta(s)}{\nu_0}\left(\frac{\kappa_0}{\kappa_f}\right)^{2s+1}\mathcal{G}_0^2 &= \frac{c}{\nu_0^{2/3}}\left(\frac{\kappa_f}{\kappa_0}\right)^{2/3}\mathcal{G}_0^{4/3} \\ \iff\ (\kappa_f/\kappa_0)^{2s+5/3} &= c\,c_\zeta(s)\,\nu_0^{-1}\mathcal{G}_0^{2/3}. \end{aligned} \tag{5.2.22}$$

Thus fixing κ_f, both (5.2.16) and (5.2.21) now read

$$N > c\,(c_\zeta(s)\nu_0^{-1}\,\mathcal{G}_0^{4s+5})^{1/(6s+5)}. \tag{5.2.23}$$

As with the case when $\bar{f}$ satisfies (3.3.1), we compare (5.2.15) and (5.2.23) to find that the bound given in (5.2.15) dominates for

$$\varepsilon M_0 \le c\,\nu_0^{3-2/(6s+5)}(c_\zeta(s)\,\mathcal{G}_0^{4s+5})^{2/(6s+5)}, \tag{5.2.24}$$

which we assume, thus giving (5.2.2).

Finally we consider $\bar{f}$ satisfying (3.3.3). Applying Lemma 9 to (3.3.6) gives

$$\int_t^{t+1} |\nabla \bar{\omega}^{>f}|^2 \,\mathrm{d}\tau \leq c\,(\varepsilon M_0)^2/\nu_0^3 + c\,\nu_0\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\,\mathcal{G}_0^2.$$

As before, the $|\nabla \bar{\omega}^{>f}|^2$ part of (5.2.14) is satisfied when both of the following hold:

$$N^2 > c\,(\varepsilon M_0)^2 \nu_0^{-5}, \qquad \text{and} \tag{5.2.25}$$

$$N^2 > c\,\nu_0^{-1}\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\,\mathcal{G}_0^2. \tag{5.2.26}$$

We equate the right hand side of (5.2.16) and (5.2.26) to obtain

$$\begin{aligned}
\frac{c}{\nu_0}\,\mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\,\mathcal{G}_0^2 &= \frac{c}{\nu_0^{2/3}}\left(\frac{\kappa_f}{\kappa_0}\right)^{2/3}\mathcal{G}_0^{4/3} \\
\iff \; (\kappa_f/\kappa_0)^{2/3}\,\mathrm{e}^{2\gamma(\kappa_f/\kappa_0-1)} &= c_\gamma^*\,\nu_0^{-1}\,\mathcal{G}_0^{2/3},
\end{aligned}$$

which we invert to find

$$\kappa_f/\kappa_0 = F_\gamma^*(\nu_0^{-1}\mathcal{G}_0^{2/3}), \tag{5.2.27}$$

where $(F_\gamma^*)^{-1}(y) := y^{2/3}\mathrm{e}^{2\gamma(y-1)}/c_\gamma^*$.

We compare (5.2.15) and (5.2.16) to determine that the bound given by (5.2.15) dominates when we assume

$$\varepsilon M_0 \leq c\left(\frac{\kappa_f}{\kappa_0}\right)^{1/3}\nu_0^{8/3}\mathcal{G}_0^{2/3}. \tag{5.2.28}$$

This gives (5.2.3). $\square$

Chapter 6

Determining nodes on the sphere

In this chapter, we state and prove our theorem concerning the number of determining nodes on the rotating sphere. One could argue that this is the most "applicable" result of this thesis, in the sense that it is based on a rotating sphere (which can be used to approximate the behaviour of the earth's atmosphere or ocean currents, for example), and requires knowledge of the relevant function at physical points, rather than its harmonics, as was with the case of modes.

In order to prove our theorem, we need a spherical analogue of Lemma 18, which requires us to choose how we allocate the nodes. We do this by triangulating the sphere, the manner of which is described in Section 6.1. The bulk of the proof of the auxiliary lemma is in Sections 6.2.2 and 6.2.3.

We begin by recalling the necessary definitions and results. From Chapter 4, we recall that the vorticity form of the Navier–Stokes equations on the sphere is given by

$$\partial_t\omega + \partial(\psi,\omega) + \frac{2}{\varepsilon}\,\partial_\phi\psi = \mu\Delta\omega + f. \tag{6.0.1}$$

Then for ω, $\omega^\sharp$ satisfying (6.0.1) with the same forcing and different initial conditions, $\delta\omega = \omega - \omega^\sharp$ satisfies

$$\partial_t\delta\omega + \partial(\psi^\sharp,\delta\omega) + \partial(\delta\psi,\omega) + \frac{2}{\varepsilon}\,\partial_\phi\delta\psi = \mu\Delta\delta\omega. \tag{6.0.2}$$

The zonal and non-zonal components of the vorticity are given by

$$\bar{\omega}(\theta,t) := \frac{1}{2\pi}\int_0^{2\pi} \omega(\theta,\phi,t)\,\mathrm{d}\phi, \qquad \text{and}$$
$$\tilde{\omega}(\theta,\phi,t) := \omega(\theta,\phi,t) - \bar{\omega}(\theta,t).$$

We also recall the idea of determining nodes from Chapter 5 and adapt it to the sphere. The set $\mathcal{E} = \{\boldsymbol{x}^1, \cdots, \boldsymbol{x}^N\} \subset S^2$ is said to be a set of determining nodes if

$$\lim_{t\to\infty} \delta\boldsymbol{v}(\boldsymbol{x}^i, t) = 0 \text{ for all } i \in \{1, \cdots, N\} \quad \text{implies} \quad \lim_{t\to\infty} |\delta\omega(t)|_{L^2} = 0.$$

6.1 Triangulation of the sphere

The implicit assumption on Lemma 18 was that the nodes were placed at equal spacings within the domain. Since we evidently cannot divide S^2 into squares the same way as in the planar case, we will need to allocate the nodes in a different manner for our spherical analogue.

Our approach is based on the icosahedral triangulation of the sphere, as follows. At iteration 0, we place $N_0 = 12$ nodes at the vertices of an inscribed regular icosahedron, and project the edges to the corresponding geodesic segments on the sphere (this is understood in what follows). The sphere is thus divided into $F_0 = 20$ equal (spherical equilateral) triangular faces. At each successive iteration, we put a new node in the middle of each edge, splitting the edge into two new edges, and connect the three new vertices on each face by new edges. The number of faces at iteration n is thus $F_n = 20 \cdot 4^n$, the number of edges is $E_n = 3F_n/2$ (since each face has three edges, each of which is shared by two faces) and by Euler's formula, the number of nodes is $N_n = 2 - F_n + E_n = 2 + 10 \cdot 4^n$.

We note that unlike the planar case, this approach implies that the faces formed by the placement of the nodes will have different sizes, for $n > 0$. In particular, the original $N_0 = 12$ nodes will always be part of exactly five neighbouring faces, whereas all other nodes will form part of six faces.

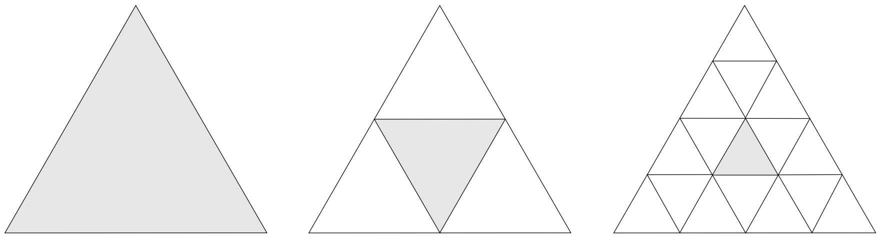

Figure 6.1: Subdivisions of a triangle at iteration n (not to scale). Shaded subtriangles are equilateral.

The triangulation also has the following property:

Lemma 21. *Any triangle Δ in the icosahedral triangulation, at any level, has corner angles that satisfy*

$$53^\circ \leq \rho_0, \rho_1, \rho_2 \leq 73^\circ. \tag{6.1.1}$$

Geometric considerations and numerical computation suggest that the sharp bounds are 72°, which is the angle of the triangles in the icosahedron, and 54°. We seek instead to show (6.1.1) in order to keep the proof simple; these could be improved to 54^- and 72^+ with some extra work but minimal conceptual difficulty. We defer the proof to the end of this chapter.

6.2 Collocation lemma

We now state the analogue of Lemma 18 for the sphere. We will use similar notation as we did in the proof of Lemma 18, in order to highlight the similarities and differences.

Lemma 22. *Let $u \in H^2(S^2)$. Define*

$$\eta(u) := \max_{1 \leq i \leq N} |u(\boldsymbol{x}^i)|,$$

where $\boldsymbol{x}^i \in S^2$, $i \in \{1, \cdots, N\}$, are the vertices of the icosahedral triangulation of S^2, as described above. Then

$$|u|^2_{L^2(S^2)} \leq c_{19}\Big(|S^2|\eta^2(u) + \frac{|S^2|^2}{N^2}|\Delta u|^2_{L^2(S^2)}\Big), \tag{6.2.1}$$

$$|\nabla u|^2_{L^2(S^2)}, |u|^2_{L^\infty(S^2)} \le c_{19}\Big(N\eta^2(u) + \frac{|S^2|}{N}|\Delta u|^2_{L^2(S^2)}\Big), \tag{6.2.2}$$

where c_{19} is an absolute constant.

We note that we have chosen to leave $|S^2|$ as it is, rather than replace with $|S^2| = 4\pi$, to show how these inequalities relate to Lemma 18 and to emphasize the (length)^2 dimension. The N here is the number of nodes N_n in the icosahedral triangulation, i.e. it takes values of $N_n = 2 + 10 \cdot 4^n$.

The proof of Lemma 22 is split into parts as follows. In Section 6.2.1, we compute bounds on "one-dimensional" norms (i.e. with one variable fixed) of an arbitrary function $v \in H^2(\Delta_*)$. Here, Δ_* is an arbitrary spherical triangle with the constraints that its (corner) angles satisfy Lemma 21, and that its sides are no larger than those of the $n = 0$ triangles (i.e. no larger than $a_0 = \pi/2 - \tan^{-1}(1/2) = 1.10714871779\cdots$). We then use these to compute an estimate for $|u|_{L^2(\Delta_*)}$ in Section 6.2.2, followed by bounds over the whole of S^2 in Section 6.2.3. We conclude in Section 6.4 by proving Lemma 21.

6.2.1 Bounds with one variable fixed

We begin by proving an intermediate result, bounding the norms of a function on Δ_* that is fixed in one variable. We first choose our coordinates such that Δ_* lies in the northern hemisphere with one of its edges along the equator $\theta = \pi/2$ and one of its nodes at $(\theta, \phi) = (\pi/2, 0)$. We denote by $\boldsymbol{c} = (\theta_c, \phi_c)$ the centroid of Δ_* and define R as the intersection of Δ_* and the spherical sector $\{(\theta, \phi) \in S^2 : \theta \le \theta_c\}$. We define h as the height of Δ_* (i.e. the geodesic distance between the top node of Δ_* and the equator), and the (left and right) longitudes where the bottom edge of R intersects Δ_* by ϕ_l and ϕ_r.

Using the spherical metric, we define the one-variable norms of v over a line of constant θ or ϕ as follows:

$$|v(\cdot, \phi')|^2_{L^2(\theta_0,\theta_1)} := \int_{\theta_0}^{\theta_1} |v(\theta, \phi')|^2 \,\mathrm{d}\theta, \tag{6.2.3}$$

$$|v(\theta', \cdot)|^2_{L^2(\phi_0,\phi_1)} := \int_{\phi_0}^{\phi_1} |v(\theta', \phi)|^2 \sin\theta' \, \mathrm{d}\phi. \tag{6.2.4}$$

However, in this chapter we only apply (6.2.4) at the equator, such that we only concern ourselves with norms of the form:

$$|v(\pi/2, \cdot)|^2_{L^2(\phi_0,\phi_1)} = \int_{\phi_0}^{\phi_1} |v(\pi/2, \phi)|^2 \, \mathrm{d}\phi. \tag{6.2.5}$$

With these assumptions, we state the following lemma, which gives an analogue of (5.1.7) and (5.1.9) for the sphere.

Lemma 23. *Let Δ_* be a triangle satisfying* (6.1.1)*, and suppose that the values of $v \in H^2(\Delta_*)$ at all corners are given. Then, for any $(\theta_p, \phi_p) \in R$, one has*

$$|v(\cdot, \phi_p)|^2_{L^2(\theta_p,\pi/2)} \le \frac{2}{|\phi_p| \sin\theta_p} |v|^2_{L^2(\Omega_p)} + \frac{|\phi_p|}{\sin\theta_p} |\partial_\phi v|^2_{L^2(\Omega_p)}, \tag{6.2.6}$$

$$|v(\pi/2, \cdot)|^2_{L^2(0,\phi_p)} \le \frac{2}{\cos\theta_p} |v|^2_{L^2(\Omega_p)} + \frac{\cos\theta_p}{\sin^2\theta_p} |\partial_\theta v|^2_{L^2(\Omega_p)}, \tag{6.2.7}$$

where $\Omega_p := [\theta_p, \pi/2] \times [0, \phi_p]$ and $L^2(\Omega_p)$ is defined using the spherical metric.

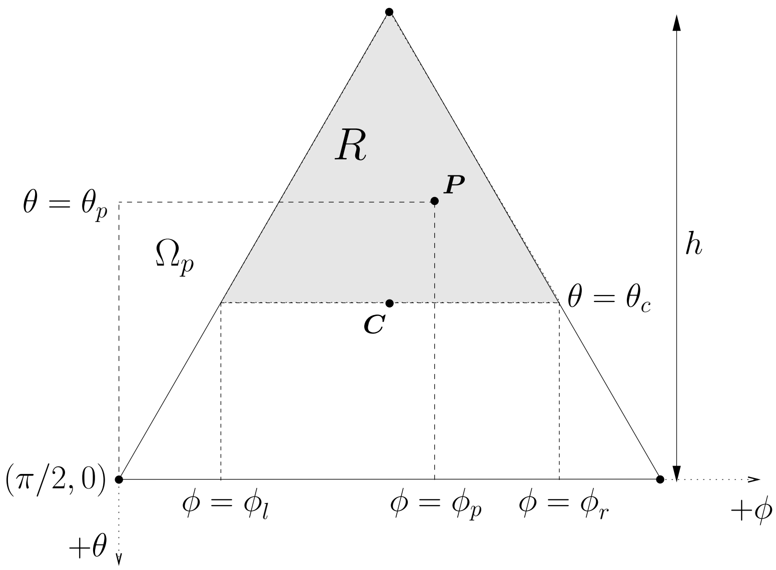

Figure 6.2: Illustration of R and Ω_p for some Δ_*.

Proof. We begin by integrating v from $(\pi/2, 0)$ to $\boldsymbol{P} = (\theta_p, \phi_p) \in R$. By the fundamental theorem of calculus,

$$v^2(\theta, \phi_p) = v^2(\theta, \phi) + \int_{\phi}^{\phi_p} \partial_\phi v^2(\theta, \phi') \, \mathrm{d}\phi', \tag{6.2.8}$$

where $(\theta, \phi) \in \Omega_p$. We integrate this with respect to θ and ϕ over Ω_p, using the spherical metric (i.e. $\mathrm{d}A = \sin\theta \; \mathrm{d}\theta \, \mathrm{d}\phi$):

$$\int_{\theta_p}^{\pi/2} \int_0^{\phi_p} v^2(\theta, \phi_p) \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta = |v|^2_{L^2(\Omega_p)} + \int_{\theta_p}^{\pi/2} \int_0^{\phi_p} \int_{\phi}^{\phi_p} \partial_\phi v^2(\theta, \phi') \, \mathrm{d}\phi' \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta. \tag{6.2.9}$$

Recalling the "fixed-ϕ" norm defined in (6.2.4), we replace $\sin\theta$ by $\sin\theta_p$ to bound the left hand side of (6.2.9) from below by

$$\begin{aligned} \int_{\theta_p}^{\pi/2} \int_0^{\phi_p} v^2(\theta, \phi_p) \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta &= |\phi_p| \int_{\theta_p}^{\pi/2} v^2(\theta, \phi_p) \sin\theta \, \mathrm{d}\theta \\ &\geq |\phi_p| \int_{\theta_p}^{\pi/2} v^2(\theta, \phi_p) \sin\theta_p \, \mathrm{d}\theta = |\phi_p| \sin\theta_p |v(\cdot, \phi_p)|^2_{L^2(\theta_p, \pi/2)}, \end{aligned} \tag{6.2.10}$$

where the absolute value of ϕ_p has been taken to emphasize that we mean the length $(\phi_p - 0)$, rather than the longitude itself. Using the spherical metric again, the right hand side of (6.2.9) is bounded by

$$\begin{aligned} |v|^2_{L^2(\Omega_p)} + &\int_{\theta_p}^{\pi/2} \int_0^{\phi_p} \int_{\phi}^{\phi_p} \partial_\phi v^2(\theta, \phi') \, \mathrm{d}\phi' \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &\leq |v|^2_{L^2(\Omega_p)} + 2 \int_{\theta_p}^{\pi/2} \int_0^{\phi_p} \int_{\phi}^{\phi_p} |v \, \partial_\phi v|(\theta, \phi') \, \mathrm{d}\phi' \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \\ &\leq |v|^2_{L^2(\Omega_p)} + 2 \int_{\phi_p}^{\pi/2} \int_0^{\phi_p} \int_0^{\phi_p} |v \, \partial_\phi v|(\theta, \phi') \sin\theta \, \mathrm{d}\phi' \, \mathrm{d}\phi \, \mathrm{d}\theta \\ &\leq |v|^2_{L^2(\Omega_p)} + 2|\phi_p| |v|_{L^2(\Omega_p)} |\partial_\phi v|_{L^2(\Omega_p)} \qquad \text{by Hölder} \\ &\leq 2|v|^2_{L^2(\Omega_p)} + |\phi_p|^2 |\partial_\phi v|^2_{L^2(\Omega_p)} \qquad \text{by Young.} \end{aligned} \tag{6.2.11}$$

We then apply (6.2.10) and (6.2.11) to (6.2.9) to obtain

$$|v(\cdot, \phi_p)|^2_{L^2(\theta_p, \pi/2)} \leq \frac{2}{|\phi_p| \sin\theta_p} |v|^2_{L^2(\Omega_p)} + \frac{|\phi_p|}{\sin\theta_p} |\partial_\phi v|^2_{L^2(\Omega_p)}, \tag{6.2.12}$$

which is (6.2.6).

In the planar case, we could obtain the y-direction analogue of the bound in the x-direction by symmetry, however we cannot do this on the sphere, due to the non-equivalence between θ and ϕ and the triangular shape of the domain. We therefore explicitly follow the steps below.

Similarly to (6.2.8), the fundamental theorem of calculus implies that

$$v^2(\pi/2,\phi) = v^2(\theta,\phi) + \int_\theta^{\pi/2} \partial_\theta v^2(\theta',\phi)\,\mathrm{d}\theta', \tag{6.2.13}$$

where $(\theta,\phi)\in\Omega_p$. Integrating this with respect to θ and ϕ over Ω_p gives

$$\int_{\theta_p}^{\pi/2}\int_0^{\phi_p} v^2(\pi/2,\phi)\sin\theta\,\mathrm{d}\phi\;\mathrm{d}\theta = |v|^2_{L^2(\Omega_p)} + \int_{\theta_p}^{\pi/2}\int_0^{\phi_p}\int_\theta^{\pi/2} \partial_\theta v^2(\theta',\phi)\,\mathrm{d}\theta'\sin\theta\,\mathrm{d}\phi\;\mathrm{d}\theta. \tag{6.2.14}$$

The left hand side of this is equal to

$$\begin{aligned}\int_{\theta_p}^{\pi/2}\int_0^{\phi_p} v^2(\pi/2,\theta)\sin\theta\;\mathrm{d}\phi\;\mathrm{d}\theta &= \int_{\theta_p}^{\pi/2}\sin\theta\int_0^{\phi_p} v^2(\pi/2,\phi)\sin(\pi/2)\;\mathrm{d}\phi\;\mathrm{d}\theta\\ &= \Big[-\cos\theta\Big]_{\theta=\theta_p}^{\theta=\pi/2}\int_0^{\phi_p} v^2(\pi/2,\phi)\sin(\pi/2)\;\mathrm{d}\phi = \cos\theta_p\,|v(\pi/2,\cdot)|^2_{L^2(0,\phi_p)}.\end{aligned} \tag{6.2.15}$$

The right hand side of (6.2.14) is bounded by

$$\begin{aligned}&|v|^2_{L^2(\Omega_p)} + \int_{\theta_p}^{\pi/2}\int_0^{\phi_p}\int_\theta^{\pi/2}\partial_\theta v^2(\theta',\phi)\;\mathrm{d}\theta'\sin\theta\;\mathrm{d}\phi\;\mathrm{d}\theta\\
&\quad\le |v|^2_{L^2(\Omega_p)} + 2\int_{\theta_p}^{\pi/2}\int_0^{\phi_p}\int_\theta^{\pi/2}|v\,\partial_\theta v|(\theta',\phi)\;\mathrm{d}\theta'\sin\theta\;\mathrm{d}\phi\;\mathrm{d}\theta\\
&\quad\le |v|^2_{L^2(\Omega_p)} + 2\int_{\theta_p}^{\pi/2}\int_0^{\phi_p}\int_{\theta_p}^{\pi/2}|v\,\partial_\theta v|(\theta',\phi)\,\frac{\sin\theta'}{\sin\theta'}\;\mathrm{d}\theta'\;\mathrm{d}\phi\;\sin\theta\;\mathrm{d}\theta\\
&\quad\le |v|^2_{L^2(\Omega_p)} + \frac{2}{\sin\theta_p}\int_{\theta_p}^{\pi/2}\int_0^{\phi_p}\int_{\theta_p}^{\pi/2}|v\,\partial_\theta v|(\theta',\phi)\sin\theta'\;\mathrm{d}\theta'\;\mathrm{d}\phi\;\sin\theta\;\mathrm{d}\theta\\
&\quad= |v|^2_{L^2(\Omega_p)} + \frac{2}{\sin\theta_p}\Big[-\cos\theta\Big]_{\theta=\theta_p}^{\theta=\pi/2}\int_{\theta_p}^{\pi/2}\int_0^{\phi_p}|v\,\partial_\theta v|(\theta',\phi)\sin\theta'\;\mathrm{d}\theta'\;\mathrm{d}\phi\\
&\quad\le |v|^2_{L^2(\Omega_p)} + 2\,\frac{\cos\theta_p}{\sin\theta_p}\,|v|_{L^2(\Omega_p)}|\partial_\theta v|_{L^2(\Omega_p)} &&\text{by Hölder}\\
&\quad\le 2|v|^2_{L^2(\Omega_p)} + \frac{\cos^2\theta_p}{\sin^2\theta_p}\,|\partial_\theta v|^2_{L^2(\Omega_p)} &&\text{by Young.}\end{aligned} \tag{6.2.16}$$

Applying (6.2.15) and (6.2.16) to (6.2.14) thus gives (6.2.7):

$$|v(\pi/2,\cdot)|^2_{L^2(0,\phi_p)} \le \frac{2}{\cos\theta_p}\,|v|^2_{L^2(\Omega_p)} + \frac{\cos\theta_p}{\sin^2\theta_p}\,|\partial_\theta v|^2_{L^2(\Omega_p)}. \tag{6.2.17}$$

□

Using this lemma, we bound $|u|_{L^2(\Delta_*)}$ in the following section.

6.2.2 Bounds over a triangle

The main part of the proof of Lemma 22 is split over this section and Section 6.2.3. We return to our function $u \in H^2(S^2)$ and derive bounds over Δ_*, which depends on values of u outside of Δ_*.

Proof of Lemma 22: We begin by integrating u from $(\pi/2, 0)$ to $(\theta, \phi) \in R \subset \Delta_*$. By the triangle inequality, we have

$$u(\theta,\phi) - u(\pi/2,0) \leq |u(\theta,\phi) - u(\pi/2,\phi)| + |u(\pi/2,\phi) - u(\pi/2,0)|,$$

which implies that

$$u^2(\theta,\phi) \leq 3u^2(\pi/2,0) + 3|u(\theta,\phi) - u(\pi/2,\phi)|^2 + 3|u(\pi/2,\phi) - u(\pi/2,0)|^2. \quad (6.2.18)$$

The second term on the right hand side of this is bounded by

$$\begin{aligned} |u(\theta,\phi) - u(\pi/2,\phi)|^2 &= \left| \int_\theta^{\pi/2} \partial_\theta u(\theta',\phi)\,\mathrm{d}\theta' \right|^2 \leq |\partial_\theta u(\cdot,\phi)|^2_{L^1(\theta,\pi/2)} \\ &\leq |1|^2_{L^2(\theta,\pi/2)} |\partial_\theta u(\cdot,\phi)|^2_{L^2(\theta,\pi/2)} \qquad \text{by Hölder} \\ &= (\pi/2 - \theta)|\partial_\theta u(\cdot,\phi)|^2_{L^2(\theta,\pi/2)} \leq |h||\partial_\theta u(\cdot,\phi)|^2_{L^2(\theta,\pi/2)} \\ &\leq |h||\partial_\theta u(\cdot,\phi)|^2_{L^2(\pi/2-h,\pi/2)}. \end{aligned} \quad (6.2.19)$$

We apply (6.2.6) to $v = \partial_\theta u$, θ_p replaced by our $\theta \in [\pi/2 - h, \theta_c]$, ϕ_p being our $\phi \in [\phi_l, \phi_r]$ and $\Omega_p = [\theta, \pi/2] \times [0, \phi]$ to bound this further by

$$\begin{aligned} |u(\theta,\phi) - u(\pi/2,\phi)|^2 &\leq |h||\partial_\theta u(\cdot,\phi)|^2_{L^2(\pi/2-h,\pi/2)} \\ &\leq |h|\Big(\frac{2}{|\phi|\sin\theta} |\partial_\theta u|^2_{L^2(\Omega_q)} + \frac{|\phi|}{\sin\theta} |\partial^2_{\theta\phi} u|^2_{L^2(\Omega_q)} \Big) \\ &\leq |h|\Big(\frac{2}{|\phi_l|\sin(\pi/2 - h)} |\partial_\theta u|^2_{L^2(\Omega_q)} + \frac{|\phi_r|}{\sin(\pi/2-h)} |\partial^2_{\theta\phi} u|^2_{L^2(\Omega_q)} \Big), \end{aligned} \quad (6.2.20)$$

where $\Omega_q := [\pi/2 - h, \pi/2] \times [0, \phi_r]$, and the last line follows by observing that $|\phi|^{-1} \leq |\phi_l|^{-1}$, $1/\sin\theta \leq 1/\sin(\pi/2 - h)$ and $|\phi| \leq |\phi_r|$.

Similarly, the third term on the right hand side of (6.2.18) can be bounded by using

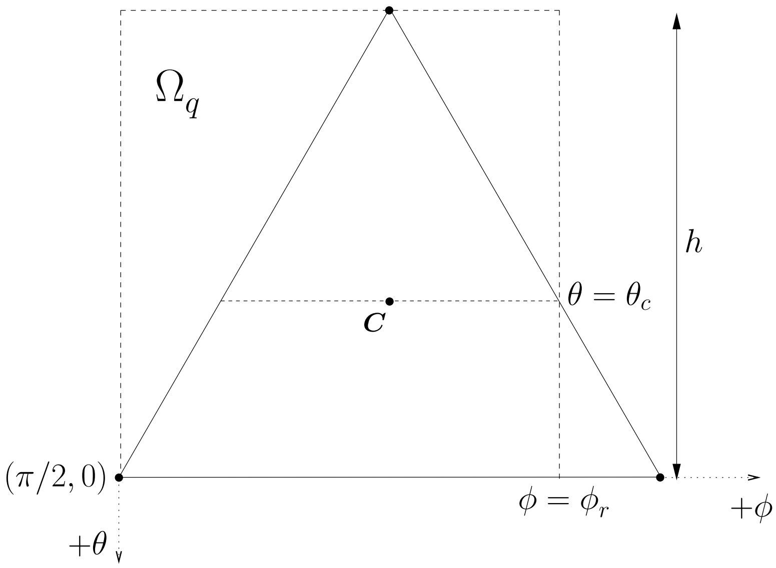

Figure 6.3: Illustration of Ω_q.

(6.2.7) on $v = \partial_\phi u$:

$$
\begin{aligned}
|u(\pi/2,\phi) - u(\pi/2,0)|^2 &= \left|\int_0^\phi \partial_\phi u(\pi/2,\phi')\sin(\pi/2)\,\mathrm{d}\phi'\right|^2 \le |\partial_\phi u(\pi/2,\cdot)|^2_{L^1(0,\phi)} \\
&\le |1|^2_{L^2(0,\phi)} |\partial_\phi u(\pi/2,\cdot)|^2_{L^2(0,\phi)} \qquad \text{by Hölder} \\
&\le |\phi_r| |\partial_\phi u(\pi/2,\cdot)|^2_{L^2(0,\phi_r)} \\
&\le |\phi_r| \Big(\frac{2}{\cos\theta} |\partial_\phi u|^2_{L^2(\Omega_q)} + \frac{\cos\theta}{\sin^2\theta} |\partial^2_{\theta\phi} u|^2_{L^2(\Omega_q)} \Big) \\
&\le |\phi_r| \Big(\frac{2}{\cos\theta_c} |\partial_\phi u|^2_{L^2(\Omega_q)} + \frac{\cos(\pi/2-h)}{\sin^2(\pi/2-h)} |\partial^2_{\theta\phi} u|^2_{L^2(\Omega_q)} \Big),
\end{aligned} \tag{6.2.21}
$$

where the last line follows from noting that $1/\cos\theta \le 1/\cos\theta_c$, $\cos\theta \le \cos(\pi/2 - h)$ and $1/\sin^2\theta \le 1/\sin^2(\pi/2 - h)$. Thus by using (6.2.20) and (6.2.21), (6.2.18) becomes

$$
\begin{aligned}
u^2(\theta,\phi) &\le 3u^2(\pi/2,0) + 3|u(\theta,\phi) - u(\pi/2,\phi)|^2 + 3|u(\pi/2,\phi) - u(\pi/2,0)|^2 \\
&\le 3u^2(\pi/2,0) + \frac{6|h|}{|\phi_l|\sin(\pi/2-h)} |\partial_\theta u|^2_{L^2(\Omega_q)} + 3\,\frac{|h||\phi_r|}{\sin(\pi/2-h)} |\partial^2_{\theta\phi} u|^2_{L^2(\Omega_q)} \\
&\quad + \frac{6|\phi_r|}{\cos\theta_c} |\partial_\phi u|^2_{L^2(\Omega_q)} + 3\,\frac{|\phi_r|\cos(\pi/2-h)}{\sin^2(\pi/2-h)} |\partial^2_{\theta\phi} u|^2_{L^2(\Omega_q)} \\
&\le 3u^2(\pi/2,0) + 6\Big(\frac{|h|}{|\phi_l|\sin(\pi/2-h)} + \frac{|\phi_r|}{\cos\theta_c} \Big) |\nabla u|^2_{L^2(\Omega_q)} \\
&\quad + 3\Big(\frac{|h||\phi_r|}{\sin(\pi/2-h)} + \frac{|\phi_r|\cos(\pi/2-h)}{\sin^2(\pi/2-h)} \Big) |\partial^2_{\theta\phi} u|^2_{L^2(\Omega_q)}.
\end{aligned} \tag{6.2.22}
$$

We integrate this over R to obtain

$$\begin{aligned}|u|^2_{L^2(R)} \leq 3|R|u^2(\pi/2,0) + 6|R|\Big(\frac{|h|}{|\phi_l|\sin(\pi/2-h)} + \frac{|\phi_r|}{\cos\theta_c}\Big)|\nabla u|^2_{L^2(\Omega_q)} \\ + 3|R|\Big(\frac{|h||\phi_r|}{\sin(\pi/2-h)} + \frac{|\phi_r|\cos(\pi/2-h)}{\sin^2(\pi/2-h)}\Big)|\partial^2_{\theta\phi}u|^2_{L^2(\Omega_q)}.\end{aligned} \tag{6.2.23}$$

Recalling the periodic case (5.1.15), in order to bound $|u|_{L^2(S^2)}$ we would like to "rotate" R twice so that the union of the three Rs corresponding to the three nodes will, by definition, cover Δ_*. We denote by R' and R'' the two other regions defined analogously to R, corresponding to the different choice of node of Δ_* being at $(\pi/2, 0)$, with $\Omega_{q'}$ and $\Omega_{q''}$ also defined similarly. Taking the sum of (6.2.23) corresponding to each node gives us a bound on u over Δ_*, keeping in mind that $\Omega_q \cup \Omega_{q'} \cup \Omega_{q''}$ is not contained within Δ_*:

$$\begin{aligned}|u|^2_{L^2(\Delta_*)} &\leq |u|^2_{L^2(R)} + |u|^2_{L^2(R')} + |u|^2_{L^2(R'')} \\ &\leq 9R_{\max}\,\eta^2(u) + 6R_{\max}\Big(\frac{|h_*|}{|\phi_{l*}|\sin(\pi/2-h_*)} + \frac{|\phi_{r*}|}{\cos\theta_{c*}}\Big) \\ &\qquad\qquad \cdot\Big(|\nabla u|^2_{L^2(\Omega_q)} + |\nabla u|^2_{L^2(\Omega_{q'})} + |\nabla u|^2_{L^2(\Omega_{q''})}\Big) \\ &\quad + 3R_{\max}\Big(\frac{|h_*||\phi_{r*}|}{\sin(\pi/2-h_*)} + \frac{|\phi_{r*}|\cos(\pi/2-h_*)}{\sin^2(\pi/2-h_*)}\Big) \\ &\qquad\qquad \cdot\Big(|\partial^2_{\theta\phi}u|^2_{L^2(\Omega_q)} + |\partial^2_{\theta\phi}u|^2_{L^2(\Omega_{q'})} + |\partial^2_{\theta\phi}u|^2_{L^2(\Omega_{q''})}\Big),\end{aligned}$$

where $R_{\max} := \max\{|R|, |R'|, |R''|\}$. The $*$ subscript denotes the (same) choice of node such that the right hand side is maximal (i.e. h_*, ϕ_{l*}, ϕ_{r*} and θ_{c*} individually may not necessarily be the largest of their corresponding parameters).

Now that we have bounds on u over Δ_*, we return to our triangulation of the sphere. We denote by Δ_i, $i \in \{1, \cdots, N = N_n\}$, the triangles covering the sphere; they can be compared to Q_i in the periodic case. For fixed $\boldsymbol{x} \in S^2$, we choose our Δ_* to be the Δ_i containing $\boldsymbol{x}$. Then Ω_q is covered by the union of at most 6 other Δ_i, i.e. $\Omega_q \subseteq \cup_{i=1}^6 \Delta_i$ (via a possible relabelling of the indices). We "rotate" R twice (i.e. define R' and R'' as before) and take the sums of their corresponding forms of

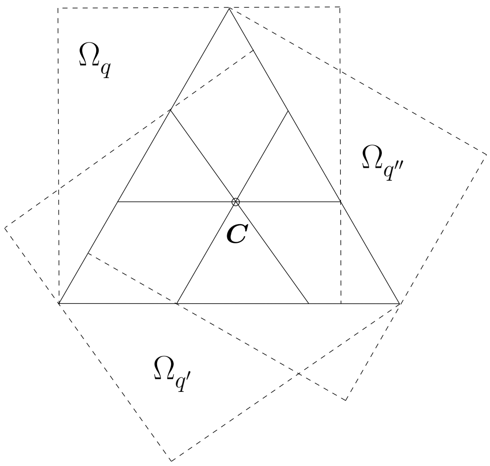

Figure 6.4: Ω_q and its "rotations" $\Omega_{q'}$, $\Omega_{q''}$, which are defined via the choice of corner node.

(6.2.23) to cover Δ_i:

$$|u|^2_{L^2(\Delta_i)} \leq 9R_{\max}\eta^2(u) + 6R_{\max}\left(\frac{|h_{i'}|}{|\phi_{li'}|\sin(\pi/2 - h_{i'})} + \frac{|\phi_{ri'}|}{\cos\theta_{ci'}}\right)|\nabla u|^2_{L^2(\cup_{k=1}^{18}\Delta_{b(k)})}$$
$$+ 3R_{\max}\left(\frac{|h_{i'}||\phi_{ri'}|}{\sin(\pi/2 - h_{i'})} + \frac{|\phi_{ri'}|\cos(\pi/2 - h_{i'})}{\sin^2(\pi/2 - h_{i'})}\right)|\partial^2_{\theta\phi}u|^2_{L^2(\cup_{k=1}^{18}\Delta_{b(k)})}, \tag{6.2.24}$$

where the i' subscript denotes the choice of node of Δ_i to integrate from, such that the whole right hand side is maximal. We thus have a bound on u over each Δ_i.

In the next section, we sum (6.2.24) over S^2 to obtain our desired bounds of (6.2.1) and (6.2.2).

6.2.3 Bounds over S^2

Now that we have a bound on u over each Δ_i, we take their sum to obtain a bound over S^2. Summing (6.2.24) over i gives

$$|u|^2_{L^2(S^2)} = |u|^2_{L^2(\cup_{i=1}^N\Delta_i)}$$
$$\leq 9|S^2|\eta^2(u) + 6\cdot 18\max_i\left\{|\Delta_i|\left(\frac{|h_{i'}|}{|\phi_{li'}|\sin(\pi/2 - h_{i'})} + \frac{|\phi_{ri'}|}{\cos\theta_{ci'}}\right)\right\}|\nabla u|^2_{L^2(S^2)}$$
$$+ 3\cdot 18\max_i\left\{|\Delta_i|\left(\frac{|h_{i'}||\phi_{ri'}|}{\sin(\pi/2 - h_{i'})} + \frac{|\phi_{ri'}|\cos(\pi/2 - h_{i'})}{\sin^2(\pi/2 - h_{i'})}\right)\right\}|\partial^2_{\theta\phi}u|^2_{L^2(S^2)}$$
$$\leq 9|S^2|\eta^2(u) + 108|\Delta_{\max}|\left(\frac{|h_M|}{|\phi_{lM}|\sin(\pi/2 - h_M)} + \frac{|\phi_{rM}|}{\cos\theta_{cM}}\right)|u|_{L^2(S^2)}|\Delta u|_{L^2(S^2)}$$

$$+ 54|\Delta_{\max}|\Big(\frac{|h_M||\phi_{rM}|}{\sin(\pi/2-h_M)}+\frac{|\phi_{rM}|\cos(\pi/2-h_M)}{\sin^2(\pi/2-h_M)}\Big)|\partial^2_{\theta\phi}u|^2_{L^2(S^2)}, \quad (6.2.25)$$

where $|\Delta_{\max}|$ is the area of the largest triangle, the $M = M(n)$ subscript denotes the choice of both triangle and node such that the right hand side is maximal, and the second line follows from applying (4.1.1) to the second term on the right hand side.

From here on, all norms are over the whole of S^2. By applying (4.1.12) to the last term, (6.2.25) becomes

$$\begin{aligned}|u|^2 \le 9|S^2|\eta^2(u) &+ 108|\Delta_{\max}|\Big(\frac{|h_M|}{|\phi_{lM}|\sin(\pi/2-h_M)}+\frac{|\phi_{rM}|}{\cos\theta_{cM}}\Big)|u||\Delta u|\\ &+ 54|\Delta_{\max}|\Big(\frac{|h_M||\phi_{rM}|}{\sin(\pi/2-h_M)}+\frac{|\phi_{rM}|\cos(\pi/2-h_M)}{\sin^2(\pi/2-h_M)}\Big)|\Delta u|^2.\end{aligned}$$

Applying Young's inequality to the second term on the right hand side gives

$$\begin{aligned}|u|^2 \le 9|S^2|\eta^2(u) &+ \frac{1}{2}|u|^2\\ &+ 1458|\Delta_{\max}|^2\Big(\frac{|h_M|}{|\phi_{lM}|\sin(\pi/2-h_M)}+\frac{|\phi_{rM}|}{\cos(\pi/2-h_M)}\Big)^2|\Delta u|^2\\ &+ 54|\Delta_{\max}|\Big(\frac{|h_M||\phi_{rM}|}{\sin(\pi/2-h_M)}+\frac{|\phi_{rM}|\cos(\pi/2-h_M)}{\sin^2(\pi/2-h_M)}\Big)|\Delta u|^2.\end{aligned}$$

Rearranging this, we obtain

$$\begin{aligned}|u|^2 &\le 18|S^2|\eta^2(u) + \Big[2916\Big(\frac{|h_M|}{|\phi_{lM}|\sin(\pi/2-h_M)}+\frac{|\phi_{rM}|}{\cos\theta_{cM}}\Big)^2+108c'_M\Big]|\Delta_{\max}|^2|\Delta u|^2\\ &\le 18|S^2|\eta^2(u) + c_M^2\frac{|S^2|^2}{N^2}\Big[2916\Big(\frac{|h_M|}{|\phi_{lM}|\sin(\pi/2-h_M)}+\frac{|\phi_{rM}|}{\cos\theta_{cM}}\Big)^2+108c'_M\Big]|\Delta u|^2\end{aligned} \quad (6.2.26)$$

which is (6.2.1), where

$$c'_M := \frac{|h_M||\phi_{rM}|}{\sin(\pi/2-h_M)}+\frac{|\phi_{rM}|\cos(\pi/2-h_M)}{\sin^2(\pi/2-h_M)} \qquad \text{and}$$

$$c_M := |\Delta_{\max}|\, N/|S^2|,$$

i.e. c_M is the ratio between the largest and average triangle areas. Clearly c'_M is decreasing in n, thus it is bounded by its value at $n = 0$. Using (6.4.25) and (6.4.26) below and explicitly computing all triangles of generations 0 to 4, we found that the worst value of $|h_M|/(|\phi_{lM}|\sin(\pi/2-h_M))$ is in fact attained at generation 0, and that of $|\phi_{rM}|/\cos\theta_{cM}$ is approached in the limit of large generation.

Applying Young's inequality to (6.2.1) will give the first part of (6.2.2):

$$\begin{aligned}
|\nabla u|_{L^2}^2 &= -(u,\Delta u) \le |u\Delta u|_{L^1} \le |u|_{L^2}|\Delta u|_{L^2} && \text{by Hölder}\\
&\le \frac{N}{2|S^2|}|u|^2 + \frac{|S^2|}{2N}|\Delta u|^2 && \text{by Young}\\
&\le c\,\frac{N}{2|S^2|}\Big(|S^2|\eta^2(u) + \frac{|S^2|^2}{N^2}|\Delta u|^2\Big) + \frac{|S^2|}{2N}|\Delta u|^2 && \text{by (6.2.1)}\\
&= c\Big(N\eta^2(u) + \frac{|S^2|}{N}|\Delta u|^2\Big).
\end{aligned}$$

Applying Agmon (2.2.6) and following essentially identical computations gives the second part of (6.2.2):

$$\begin{aligned}
|u|_\infty^2 &\le c\,|u|_{L^2}|\Delta u|_{L^2}\\
&\le \cdots \le c\Big(N\eta^2(u) + \frac{|S^2|}{N}|\Delta u|^2\Big).
\end{aligned}$$

To finish the proof, we prove Lemma 21 in Section 6.4, which immediately implies that the right hand side of (6.2.26) can be bounded independently of the number of iterations n.

6.3 Bounds on the number of determining nodes

We now state and prove our main result. As with our proof on the modes, we consider different forms of the zonal forcing $\bar{f}$, as given in (4.3.1) to (4.3.3), and the consequences they have on the number of nodes.

Theorem 24 (Determining nodes on the sphere)**.** *Let $\delta\omega$ be the solution of* (6.0.2) *with $f \in H^2(S^2)$. Then there exists a set of determining nodes $\mathcal{E} = \{\boldsymbol{x}^1, \cdots, \boldsymbol{x}^N\} \subset S^2$ when*

(a) $\bar{f}$ satisfies (4.3.1) *and*

$$N > c_{20}(\nu_0)\,\max\Big\{\varepsilon^{1/2}M_0^{1/2}, (\kappa_f/\kappa_0)^{1/3}\,\mathcal{G}_0^{2/3}\Big\},\ \ \textit{or} \tag{6.3.1}$$

(b) $\bar f$ satisfies (4.3.2) *and*

$$N > c_{21}(\nu_0, s)\ \max\Big\{\varepsilon^{1/2} M_0^{1/2},\ \mathcal{G}_0^{(4s+5)/(6s+5)}\Big\}\ \text{or} \tag{6.3.2}$$

(c) $\bar f$ satisfies (4.3.3) *and*

$$N > c_{22}(\nu_0)\ \max\Big\{\varepsilon^{1/2} M_0^{1/2},\ F^*_{\gamma'}(\nu_0^{-1}\,\mathcal{G}_0^{2/3})^{1/3}\,\mathcal{G}_0^{2/3}\Big\}, \tag{6.3.3}$$

for constants c_{20}, c_{21}, c_{22}, $F^*_{\gamma'}$ defined as in (6.3.27) *below and sufficiently small ε.*

The above-mentioned smallness requirements on ε are given in (6.3.17), (6.3.23) and (6.3.28). We have chosen to state them later and separately from the statement of the theorem for clarity and simplicity; one could include them here in exchange for longer and more complicated bounds on N.

Proof. The proof is essentially identical to that of the planar case, but using the spherical estimates we have derived in place of their planar analogues. We begin by multiplying (6.0.2) by $\delta\omega$ in L^2:

$$(\partial_t\delta\omega, \delta\omega) + (\partial(\psi^\sharp, \delta\omega), \delta\omega) + (\partial(\delta\psi, \omega), \delta\omega) + \frac{2}{\varepsilon}(\partial_\phi\delta\psi, \delta\omega) = \mu(\Delta\delta\omega, \delta\omega). \tag{6.3.4}$$

As in (4.4.5), the fourth term is 0 by the antisymmetric property of $\partial_\phi\Delta^{-1}$:

$$\frac{2}{\varepsilon}(\partial_\phi\delta\psi, \delta\omega) = \frac{2}{\varepsilon}(\partial_\phi\delta\psi, \Delta\delta\psi) = -\frac{2}{\varepsilon}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathrm{i}m\, l(l+1)\,\delta\psi_{lm}\overline{\delta\psi_{lm}} = 0.$$

The second term of (6.3.4) is also 0, due to (4.1.4). Thus (6.3.4) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega|^2 + (\partial(\delta\psi, \omega), \delta\omega) = \mu(\Delta\delta\omega, \delta\omega) = -\mu|\nabla\delta\omega|^2. \tag{6.3.5}$$

By splitting the vorticity into $\omega = \bar\omega + \tilde\omega$, we expand the second term of (6.3.5) as

$$(\partial(\delta\psi, \omega), \delta\omega) = (\partial(\delta\psi, \bar\omega), \delta\omega) + (\partial(\delta\psi, \tilde\omega), \delta\omega). \tag{6.3.6}$$

We further split the zonal vorticity $\bar\omega$ into $\bar\omega = \bar\omega^{<f} + \bar\omega^{>f}$, where we recall $\bar\omega^{<f} = \mathsf{P}_{\kappa_f}\bar\omega$ and $\bar\omega^{>f} = \bar\omega - \bar\omega^{<f}$, for some $\kappa_f \ge \kappa_0$ that we fix later. We thus rearrange (6.3.5):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega|^2 + \mu|\nabla\delta\omega|^2 = -(\partial(\delta\psi, \tilde\omega), \delta\omega) - (\partial(\delta\psi, \bar\omega^{<f}), \delta\omega) - (\partial(\delta\psi, \bar\omega^{>f}), \delta\omega). \tag{6.3.7}$$

As mentioned earlier, we choose $\mathcal{E}$ to be the set of N vertices of the icosahedral triangulation of S^2. We then apply (6.2.2) to bound the first term on the right hand side of (6.3.7):

$$\begin{aligned}
|(\partial(\delta\psi,\tilde{\omega}),\delta\omega)| &\le |\nabla\delta\psi|_\infty|\nabla\tilde{\omega}|_2|\delta\omega|_2 && \text{by Hölder}\\
&\le c\,\frac{\mu N}{|S^2|}|\nabla\delta\psi|_\infty^2 + c\,\frac{|S^2|}{\mu N}|\nabla\tilde{\omega}|^2|\delta\omega|^2 && \text{by Young}\\
&\le c\,\frac{\mu N}{|S^2|}\Big[N\eta^2(\nabla\delta\psi)+\frac{|S^2|}{N}|\nabla\delta\omega|^2\Big] + c\,\frac{|S^2|}{\mu N}|\nabla\tilde{\omega}|^2|\delta\omega|^2 && \text{by (6.2.2).}
\end{aligned} \tag{6.3.8}$$

Next, we apply (6.2.1) on the second term on the right hand side of (6.3.7):

$$\begin{aligned}
|(\partial(\delta\psi,\bar{\omega}^{<f}),\delta\omega)| &\le |\nabla\delta\psi|_2|\nabla\bar{\omega}^{<f}|_\infty|\delta\omega|_2\\
&\le c\,\kappa_0^{1/2}|\nabla\delta\psi||\nabla\bar{\omega}^{<f}|^{1/2}|\Delta\bar{\omega}^{<f}|^{1/2}|\delta\omega| && \text{by Agmon}\\
&\le c\,(\kappa_0\kappa_f)^{1/2}|\nabla\omega||\nabla\delta\psi||\delta\omega| && \text{by (4.2.14)}\\
&\le c\,\frac{\mu N^2}{|S^2|^2}|\nabla\delta\psi|^2 + c\,\frac{|S^2|^2}{\mu N^2}\,\kappa_0\kappa_f|\nabla\omega|^2|\delta\omega|^2 && \text{by Young}\\
&\le c\,\frac{\mu N^2}{|S^2|^2}\Big[|S^2|\eta^2(\nabla\delta\psi)+\frac{|S^2|^2}{N^2}|\nabla\delta\omega|^2\Big] + c\,\frac{|S^2|^2}{\mu N^2}\,\kappa_0\kappa_f|\nabla\omega|^2|\delta\omega|^2 && \text{by (6.2.1).}
\end{aligned} \tag{6.3.9}$$

We bound the last term of (6.3.7) as

$$\begin{aligned}
|(\partial(\delta\psi,\bar{\omega}^{>f}),\delta\omega)| &\le |\nabla\delta\psi|_\infty|\nabla\bar{\omega}^{>f}|_2|\delta\omega|_2\\
&\le c\,\frac{\mu N}{|S^2|}|\nabla\delta\psi|_\infty^2 + c\,\frac{|S^2|}{\mu N}|\nabla\bar{\omega}^{>f}|^2|\delta\omega|^2 && \text{by Young}\\
&\le c\,\frac{\mu N}{|S^2|}\Big[N\eta^2(\nabla\delta\psi)+\frac{|S^2|}{N}|\nabla\delta\omega|^2\Big] + c\,\frac{|S^2|}{\mu N}|\nabla\bar{\omega}^{>f}|^2|\delta\omega|^2 && \text{by (6.2.2).}
\end{aligned} \tag{6.3.10}$$

Finally, we apply (6.2.2) to the second term on the left hand side of (6.3.7) and rearrange to obtain

$$\begin{aligned}
&|\delta\omega|^2 \le c\Big(N\eta^2(\nabla\delta\psi)+\frac{|S^2|}{N}|\nabla\delta\omega|^2\Big)\\
\iff\quad & c\,\frac{\mu N}{|S^2|}|\delta\omega|^2 - c\,\frac{\mu N^2}{|S^2|}\,\eta^2(\nabla\delta\psi) \le \mu|\nabla\delta\omega|^2.
\end{aligned} \tag{6.3.11}$$

Thus putting together (6.3.7) to (6.3.11) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|\delta\omega|^2 + |\delta\omega|^2\Big[c\,\frac{\mu N}{|S^2|} - c\,\frac{|S^2|}{\mu N}|\nabla\tilde{\omega}|^2 - c\,\frac{|S^2|^2}{\mu N^2}\,\kappa_0\kappa_f|\nabla\omega|^2 - c\,\frac{|S^2|}{\mu N}|\nabla\bar{\omega}^{>f}|^2\Big]$$
$$\le c\,\frac{\mu N^2}{|S^2|}\,\eta^2(\nabla\delta\psi). \tag{6.3.12}$$

We seek to apply Lemma 8 to $\xi = |\delta\omega|^2$, ρ being the bracket on the left hand side and σ the right hand side of (6.3.12). The hypothesis of the lemma on $\sigma = (c\mu N^2/|S^2|)\,\eta^2(\nabla\delta\psi)$ is met because $\nabla\delta\psi(\boldsymbol{x}^i,t) \to 0$ as $t \to \infty$ for all i (by construction) and $|\nabla\omega|$ is bounded due to (4.4.20), while that on ξ follows from the regularity of the Navier–Stokes equations.

The hypothesis on ρ would follow immediately if

$$\limsup_{t\to\infty}\int_t^{t+1}\Big(\frac{|S^2|}{\mu N}|\nabla\tilde{\omega}|^2 + \frac{|S^2|^2}{\mu N^2}\,\kappa_0\kappa_f|\nabla\omega|^2 + \frac{|S^2|}{\mu N}|\nabla\bar{\omega}^{>f}|^2\Big)\,\mathrm{d}\tau < c\,\frac{\mu N}{|S^2|},$$

which is equivalent to

$$\limsup_{t\to\infty}\int_t^{t+1}\Big(\frac{1}{\nu_0 N}|\nabla\tilde{\omega}|^2 + \frac{1}{\nu_0 N}\frac{\kappa_f}{\kappa_0}|\nabla\omega|^2 + \frac{1}{\nu_0 N}|\nabla\bar{\omega}^{>f}|^2\Big)\,\mathrm{d}\tau < c\,\nu_0 N, \tag{6.3.13}$$

where we recall $\nu_0 = \mu\kappa_0^2$. In turn, this follows when each of the three terms on the left hand side satisfies the inequality independently.

For the first term, we recall (3.2.9), which implies that

$$\int_t^{t+1}|\nabla\tilde{\omega}|^2\,\mathrm{d}\tau \le \varepsilon M_0/\nu_0,$$

so (6.3.13) for the $|\nabla\tilde{\omega}|^2$ term would be satisfied for

$$N^2 > c\,\varepsilon M_0/\nu_0^3. \tag{6.3.14}$$

For the second term on the left hand side of (6.3.13), we recall that (4.4.20) implies

$$\int_t^{t+1}|\nabla\omega|^2\,\mathrm{d}\tau \le c\,\nu_0\,\mathcal{G}_0^2,$$

so the $|\nabla\omega|$ part of (6.3.13) is implied by

$$N > \frac{c}{\nu_0^{1/3}}\Big(\frac{\kappa_f}{\kappa_0}\Big)^{1/3}\mathcal{G}_0^{2/3}. \tag{6.3.15}$$

For the inequality involving the $|\nabla\bar{\omega}^{>f}|^2$ term in (6.3.13), we need to handle the cases separately according to the different forms of $\bar{f}$.

First we consider when $\bar{f}$ satisfies (4.3.1). By (4.3.5),

$$\int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2 \,\mathrm{d}\tau \le c\,\varepsilon^2 M_0^2/\nu_0^3,$$

so the $|\nabla\bar{\omega}^{>f}|$ part of (6.3.13) holds if

$$N > c\,\varepsilon M_0/\nu_0^{5/2}. \tag{6.3.16}$$

This bound is dominated by (6.3.14) when

$$\varepsilon M_0 \le c\,\nu_0^2. \tag{6.3.17}$$

Upon assuming this, (6.3.1) follows from (6.3.14) and (6.3.15).

For $\bar{f}$ instead satisfying (4.3.2), we apply Lemma 9 to (3.3.5) to observe that

$$\int_t^{t+1} |\nabla\bar{\omega}^{>f}|^2 \,\mathrm{d}\tau \le c\,\varepsilon^2 M_0^2/\nu_0^3 + c\,c_\zeta(s)\,\nu_0\,(\kappa_0/\kappa_f)^{2s+1}\,\mathcal{G}_0^2 = I_1, \tag{6.3.18}$$

where $1/c_\zeta(s) = (2s+1)\,\zeta(2s+2)$. Thus the $|\nabla\bar{\omega}^{>f}|^2$ part of (6.3.13) would be satisfied if $I_1 \le c\,\nu_0^2 N^2$; analogously to what we did with (6.3.13), this in turn is implied by

$$N^2 > c\,(\varepsilon M_0)^2/\nu_0^5, \tag{6.3.19}$$

$$N^2 > \frac{c}{\nu_0}\,c_\zeta(s)(\kappa_0/\kappa_f)^{2s+1}\,\mathcal{G}_0^2. \tag{6.3.20}$$

Since (6.3.15) and (6.3.20) must both hold, we equate these bounds to find the value of κ_f that minimises the bounds on N:

$$c\,\frac{c_\zeta(s)}{\nu_0}\left(\frac{\kappa_0}{\kappa_f}\right)^{2s+1}\mathcal{G}_0^2 = \frac{c}{\nu_0^{2/3}}\left(\frac{\kappa_f}{\kappa_0}\right)^{2/3}\mathcal{G}_0^{4/3}$$

$$\iff (\kappa_f/\kappa_0)^{2s+5/3} = c\,c_\zeta(s)\,\nu_0^{-1}\mathcal{G}_0^{2/3}. \tag{6.3.21}$$

Thus fixing κ_f, both (6.3.15) and (6.3.20) now become

$$N > c\,(c_\zeta(s)\nu_0^{-1}\,\mathcal{G}_0^{4s+5})^{1/(6s+5)}. \tag{6.3.22}$$

As before, we compare (6.3.14) and (6.3.22) to observe that the bound given in (6.3.14) dominates for

$$\varepsilon M_0 \leq c\, \nu_0^{3-2/(6s+5)} (c_\zeta(s)\, \mathcal{G}_0^{4s+5})^{2/(6s+5)}, \tag{6.3.23}$$

which we assume, giving (6.3.2).

Finally, we consider $\bar{f}$ satisfying (4.3.3). By applying Lemma 9 to (4.3.7), we note that

$$\int_t^{t+1} |\nabla \bar{\omega}^{>f}|^2 \,\mathrm{d}\tau \leq c\,(\varepsilon M_0)^2/\nu_0^3 + c\,\nu_0\, \mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\, \mathcal{G}_0^2.$$

The $|\nabla \bar{\omega}^{>f}|^2$ part of (6.3.13) is satisfied when both of the following are satisfied:

$$N^2 > c\,(\varepsilon M_0)^2/\nu_0^5, \tag{6.3.24}$$

$$N^2 > \frac{c}{\nu_0}\, \mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\, \mathcal{G}_0^2. \tag{6.3.25}$$

In order to obtain the optimal κ_f that minimises our bound on N, we equate the right hand side of (6.3.15) and (6.3.25), which leads to

$$\begin{aligned} &\frac{c}{\nu_0}\, \mathrm{e}^{2\gamma(1-\kappa_f/\kappa_0)}\, \mathcal{G}_0^2 = \frac{c}{\nu_0^{2/3}} \left(\frac{\kappa_f}{\kappa_0}\right)^{2/3} \mathcal{G}_0^{4/3} \\ \iff\ & (\kappa_f/\kappa_0)^{2/3}\, \mathrm{e}^{2\gamma(\kappa_f/\kappa_0 - 1)} = c^*_{\gamma'}\, \nu_0^{-1}\, \mathcal{G}_0^{2/3}. \end{aligned} \tag{6.3.26}$$

We invert this, which gives

$$\kappa_f/\kappa_0 = F^*_{\gamma'}(\nu_0^{-1} \mathcal{G}_0^{2/3}), \tag{6.3.27}$$

where $(F^*_{\gamma'})^{-1}(y) := y^{2/3} \mathrm{e}^{2\gamma(y-1)}/c^*_{\gamma'}$.

We compare (6.3.14) and (6.3.15) to conclude that the bound given by (6.3.14) dominates when we assume

$$\varepsilon M_0 \leq c \left(\frac{\kappa_f}{\kappa_0}\right)^{1/3} \nu_0^{8/3} \mathcal{G}_0^{2/3}. \tag{6.3.28}$$

This gives (6.3.3). □

6.4 Proof of Lemma 21

Proof of Lemma 21: Our methodology is as follows. We begin with the $n = 5$ triangulation and fix one arbitrary "ancestor" triangle $\Delta := \Delta^{(5)}$, and consider all its "descendants". To reduce clutter, we denote the sides of Δ by a, b and c (the latter does not denote a generic constant, for this proof only), with corresponding angles α, β and γ. Direct numerical computation shows that

$$0.928\cdots\frac{|\Delta^{(0)}|}{4^5} < |\Delta^{(5)}| < 1.206\cdots\frac{|\Delta^{(0)}|}{4^5} \qquad \text{and} \tag{6.4.1}$$

$$1.000\cdots\frac{a^{(0)}}{2^5} < a, b, c < 1.195\cdots\frac{a^{(0)}}{2^5}, \tag{6.4.2}$$

where $\Delta^{(0)}$ denotes the original $n = 0$ triangle with sides $a^{(0)}$. Since $|\Delta| < 10^{-3}$ is already very small, we will prove that any level n descendant $\Delta^{(n)}$ with sides $a^{(n)}_{(j)}$ satisfies

$$|\Delta^{(n)}| \simeq 4^{5-n}|\Delta|, \tag{6.4.3}$$

$$a^{(n)}_{(j)} \simeq 2^{5-n}\, a_{(j)}, \tag{6.4.4}$$

to a very good approximation, to be made precise below.

We consider a level 6 descendant of Δ with corner angle γ and adjacent sides $a/2$ and $b/2$. We denote the inner side by $c_\dagger$.

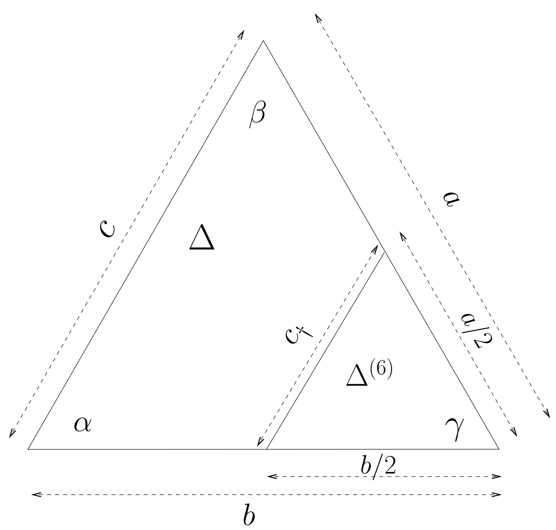

Figure 6.5: Illustration to show relationship between Δ and $\Delta^{(6)}$.

We start by considering the areas of Δ and $\Delta^{(6)}$, which are given by

$$\tan(|\Delta|/2) = \frac{\tan(a/2)\tan(b/2)\sin\gamma}{1+\tan(a/2)\tan(b/2)\cos\gamma}, \tag{6.4.5}$$

$$\tan(|\Delta^{(6)}|/2) = \frac{\tan(a/4)\tan(b/4)\sin\gamma}{1+\tan(a/4)\tan(b/4)\cos\gamma}, \tag{6.4.6}$$

and seek to prove that $|\Delta^{(6)}| \simeq |\Delta|/4$.

We note that, for $n = 5$,

$$a > b/2, \quad b > a/2, \qquad \text{and} \tag{6.4.7}$$

$$\sin\gamma > 2/3. \tag{6.4.8}$$

It is readily apparent below that these hold for all subsequent iterations.

Using Taylor's theorem for tan, $\tan^{-1}$ and sin, we write (6.4.5) and (6.4.6) as

$$|\Delta| = \frac{ab}{2}\sin\gamma + \frac{3}{5}\,(ab\sin\gamma)^2\varphi(\varrho), \tag{6.4.9}$$

$$|\Delta^{(6)}| = \frac{ab}{8}\sin\gamma + \frac{1}{30}\,(ab\sin\gamma)^2\varphi(\varrho), \tag{6.4.10}$$

where the sides a, b and c satisfy (6.4.7) and (6.4.8) and φ denotes an arbitrary function, which may change from one use to the next, such that $|\varphi(\varrho)| \leq 1$. Since $|\Delta^{(6)}|$ is very closely approximated by $|\Delta|/4$, we compare these to obtain

$$\begin{aligned} |\Delta^{(6)}| &= \frac{|\Delta|}{4} + \frac{3}{20}\,(ab\sin\gamma)^2\,\varphi(\varrho) = \frac{|\Delta|}{4} + \frac{3}{5}|\Delta|^2\varphi(\varrho) \\ &= \frac{|\Delta|}{4}\Big(1 + 2.4\,|\Delta|\varphi(\varrho)\Big). \end{aligned} \tag{6.4.11}$$

Now we consider the inner side $c_\dagger$ of $\Delta^{(6)}$. We recall the cosine rule:

$$\cos c = \cos a\cos b + \sin a\sin b\cos\gamma. \tag{6.4.12}$$

Similarly to our approach on the estimates of the areas, we use Taylor's theorem on cos to approximate $c_\dagger$ by $c/2$. Applying (6.4.12) to $\Delta^{(6)}$ gives

$$\begin{aligned} \cos c_\dagger &= \cos(a/2)\cos(b/2) + \sin(a/2)\sin(b/2)\cos\gamma \\ &= \Big(1 - \frac{1}{2}(a/2)^2 + \frac{1}{4!}(a/2)^4\varphi(\varrho)\Big)\Big(1 - \frac{1}{2}(b/2)^2 + \frac{1}{4!}(b/2)^4\varphi(\varrho)\Big) \end{aligned}$$

$$
\begin{aligned}
&+\Big(a/2+\frac{1}{6}(a/2)^3\varphi(\varrho)\Big)\Big(b/2+\frac{1}{6}(b/2)^3\varphi(\varrho)\Big)\cos\gamma \\
&=1-\frac{1}{2}\Big((a/2)^2+(b/2)^2-\frac{1}{2}\,ab\cos\gamma\Big)+12\Big(\frac{1}{8}\,ab\sin\gamma\Big)^2\varphi(\varrho), \qquad (6.4.13)
\end{aligned}
$$

where we have used the assumptions of (6.4.7) and (6.4.8) to simplify the expression. We expand the left hand side of (6.4.13) as

$$
\cos c_\dagger = 1-\frac{1}{2}\,c_\dagger^2+\frac{1}{4!}\,c_\dagger^4\,\varphi(\varrho). \qquad (6.4.14)
$$

Equating (6.4.13) and (6.4.14) thus gives

$$
\begin{aligned}
c_\dagger^2+\frac{1}{12}\,c_\dagger^4\,\varphi(\varrho) &= (a/2)^2+(b/2)^2-\frac{ab}{2}\cos\gamma+24\left(\frac{1}{8}\,ab\sin\gamma\right)^2\varphi(\varrho) \\
&= (c/2)^2+24\,|\Delta^{(6)}|^2\,\varphi(\varrho), \qquad (6.4.15)
\end{aligned}
$$

where we have used the planar cosine rule and the fact that $|\Delta^{(6)}| \geq (ab\sin\gamma)/8$ (being the area of the planar triangle with the same a, b and γ) for the second line. This expression clearly has 4 real roots in $c_\dagger$; discarding the 2 negative roots, we consider the remaining 2 possible solutions. Denoting by g the right hand side of (6.4.15), the roots of (6.4.15) are approximately at

$$
c_\dagger^2 \approx g \text{ or } \frac{-1+\sqrt{1-(g/3)\varphi(\varrho)}}{\varphi(\varrho)/6}, \qquad (6.4.16)
$$

since $c_\dagger \ll 1$. The second possible solution of $c_\dagger^2$ is, however, incompatible with $c_\dagger > 0$, so the only feasible solution is that $c_\dagger \approx \sqrt{g}$.

Returning to (6.4.15), rearranging the expression gives

$$
\begin{aligned}
c_\dagger^2\left(1+\frac{12}{5}\,|\Delta^{(6)}|^2\,\varphi(\varrho)\right) &= (c/2)^2\left(1+\frac{1}{10}\,c_\dagger^2\,\varphi(\varrho)\right)+24|\Delta^{(6)}|^2\,\varphi(\varrho) \\
&= (c/2)^2\Big(1+\frac{3}{10}\,|\Delta|\,\varphi(\varrho)\Big)+24\cdot 2|\Delta^{(6)}|\,(c/2)^2\,\varphi(\varrho) \\
&= (c/2)^2\left(1+\frac{6}{5}\,|\Delta^{(6)}|\,\varphi(\varrho)\right)+48|\Delta^{(6)}|\,(c/2)^2\,\varphi(\varrho)
\end{aligned}
\qquad (6.4.17)
$$

by assuming $c_\dagger > a/4, b/4$ and a, $b > c_\dagger$. Rearranging this further results in

$$
c_\dagger = \frac{c}{2}\Big(1+50|\Delta^{(6)}|\varphi(\varrho)\Big)^{1/2}. \qquad (6.4.18)
$$

Thus $c_\dagger$ is well approximated by $c/2$.

Obviously, analogous expressions to (6.4.11) and (6.4.18) apply to the two other corner descendants. Temporarily denoting the corner descendants by $\Delta_\alpha^{(6)}$, $\Delta_\beta^{(6)}$, $\Delta_\gamma^{(6)}$ and the central descendant by $\Delta_\zeta^{(6)}$, we have

$$|\Delta_\zeta^{(6)}| = |\Delta| - |\Delta_\alpha^{(6)}| - |\Delta_\beta^{(6)}| - |\Delta_\gamma^{(6)}|. \tag{6.4.19}$$

Thus we have also

$$|\Delta_\zeta^{(6)}| = \frac{|\Delta|}{4}\Big(1 + 2.4|\Delta|\,\varphi(\varrho)\Big), \tag{6.4.20}$$

and bounds analogous to (6.4.18) also hold for the sides of $\Delta_\zeta^{(6)}$. Thus (6.4.11) and (6.4.18) hold for all four descendants of Δ. We also note that

$$1 + 2.4|\Delta|\,\varphi(\varrho) < \frac{4}{3}. \tag{6.4.21}$$

By induction, we have

$$\begin{aligned}
|\Delta^{(n)}| &= \frac{|\Delta^{(n-1)}|}{4}\Big(1 + 2.4|\Delta^{(n-1)}|\varphi(\varrho)\Big) \\
&= \frac{|\Delta|}{4^{n-5}}\Big(1 + 2.4|\Delta^{(n-1)}|\varphi(\varrho)\Big)\cdots\Big(1 + 2.4|\Delta|\varphi(\varrho)\Big) &(6.4.22)\\
&\le \frac{|\Delta|}{3^{n-5}}. &(6.4.23)
\end{aligned}$$

We shall make use of both these bounds below. Similarly, the sides of $\Delta^{(n)}$ satisfy, with the obvious notation,

$$\begin{aligned}
a^{(n)} &= \frac{a}{2^{n-5}}\Big(1 + 50|\Delta^{(n)}|\varphi(\varrho)\Big)^{1/2}\cdots\Big(1 + 50|\Delta^{(6)}|\varphi(\varrho)\Big)^{1/2}, \\
b^{(n)} &= \frac{b}{2^{n-5}}\Big(1 + 50|\Delta^{(n)}|\varphi(\varrho)\Big)^{1/2}\cdots\Big(1 + 50|\Delta^{(6)}|\varphi(\varrho)\Big)^{1/2}, \\
c^{(n)} &= \frac{c}{2^{n-5}}\Big(1 + 50|\Delta^{(n)}|\varphi(\varrho)\Big)^{1/2}\cdots\Big(1 + 50|\Delta^{(6)}|\varphi(\varrho)\Big)^{1/2}. &(6.4.24)
\end{aligned}$$

As for the products on the right hand side, we have the upper bound

$$\begin{aligned}
&\Big(1 + 50|\Delta^{(n)}|\varphi(\varrho)\Big)^{1/2}\cdots\Big(1 + 50|\Delta^{(6)}|\varphi(\varrho)\Big)^{1/2} \\
&\qquad\le \exp\Big(25\sum_{j=6}^{n}|\Delta^{(j)}|\Big) \le \exp\Big(12.5|\Delta|\Big), &(6.4.25)
\end{aligned}$$

where for the last step we have used (6.4.23). Using the fact that $\mathrm{e}^{-2x} < 1 - x$ for $|x| < 0.5$, we find the lower bound

$$\begin{aligned}\Big(1+50|\Delta^{(n)}|\varphi(\varrho)\Big)^{1/2}\cdots\Big(1+50|\Delta^{(6)}|\varphi(\varrho)\Big)^{1/2}\\ \geq \exp\Big(-50\sum_{j=6}^{n}|\Delta^{(j)}|\Big) \geq \exp\Big(-25|\Delta|\Big).\end{aligned} \tag{6.4.26}$$

We recall that $|\Delta| \leq 10^{-3}$. This concludes the proof of Lemma 21. $\square$

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